Picsel

UE Quantum Optics

Quantum states of light:
Photon-photon correlations and Characteristic functions

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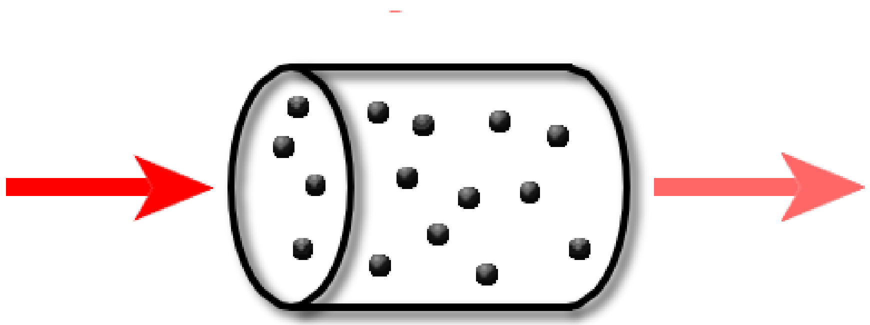
Light-Matter interactions

Quantum optics most commonly considers light interacting with individual `Quantum' systems (atoms, molecules, Quantum dots, meta-atoms ,…)

Overview : light interacting with particles (atoms)



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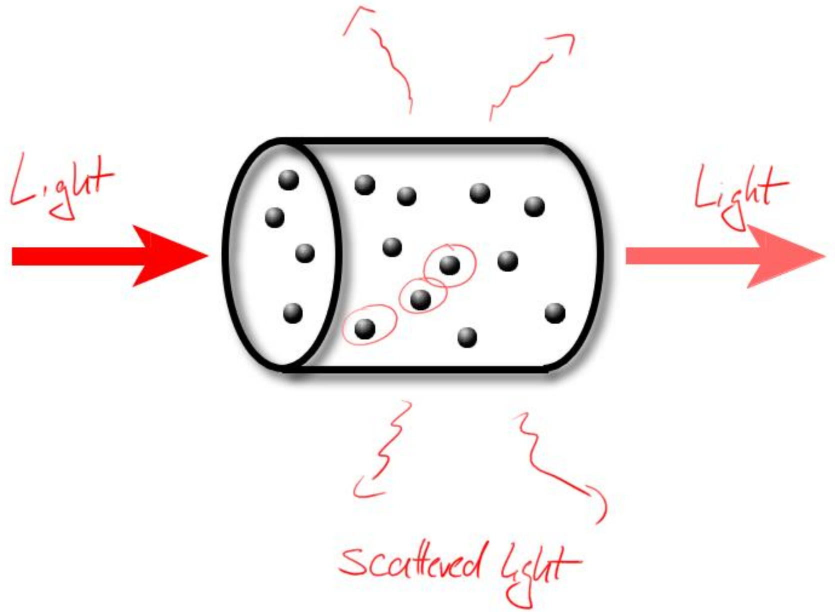
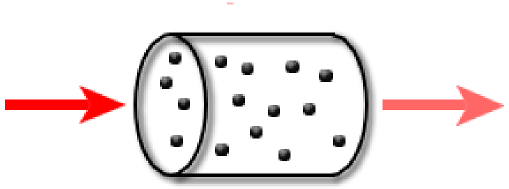
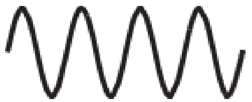
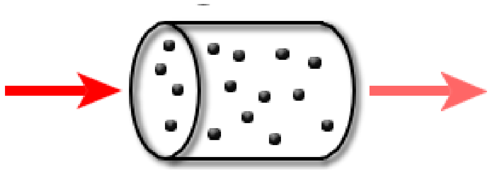


Overview



Model	Atom		Light	
a) classical	Atom classical oscillator (Hertz Dipole)		classical e.m. wave	

Overview



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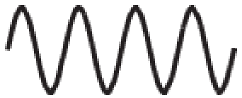
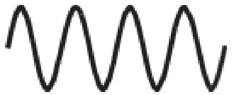
a) classical	Atom classical oscillator (Hertz Dipole)		classical e.m. wave	
b) semiclassical	Atom quantized		classical e.m. wave	

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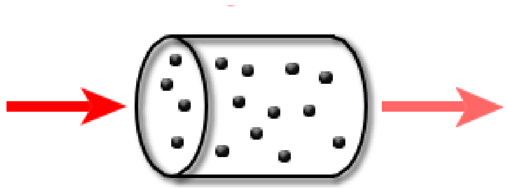

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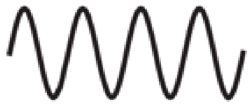
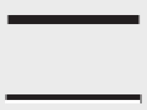
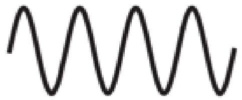
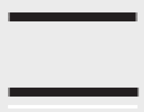
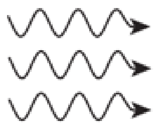
a) classical	Atom classical oscillator (Hertz Dipole)		classical e.m. wave	
b) semiclassical	Atom quantized		classical e.m. wave	
c) quantum mechanical	Atom quantized		quantized field	

Limitations of Models

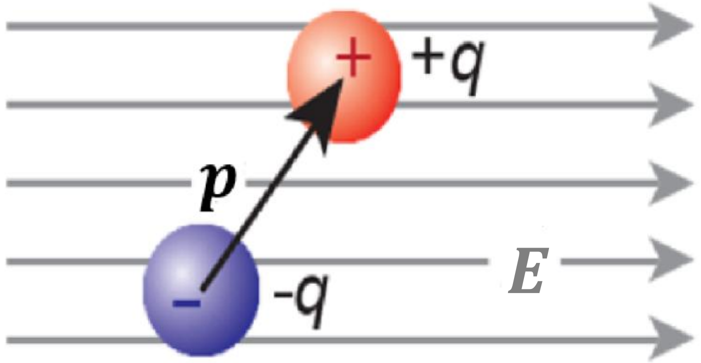
a) classical oscillating field drives classical electron

- ✔ absorption, dispersion
- ✖ black body radiation, non-linear effects

b) semiclassical description

- ✔ all of the above
- ✖ photon statistics, non-classical light fields

Light-atom Interaction

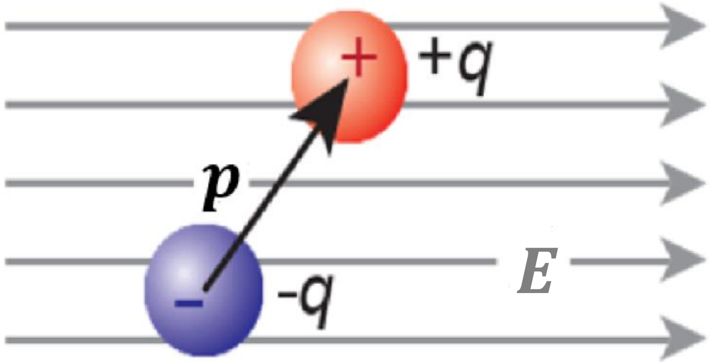


a) Classical dipole in an electric field

Dipole moment : $\boldsymbol{p} = q\boldsymbol{r}$

Interaction potential energy : $U_I = -\boldsymbol{p} \cdot \boldsymbol{E}$

Light-atom Interaction Hamiltonian

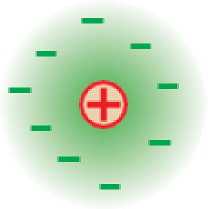


a) Classical dipole in an electric field

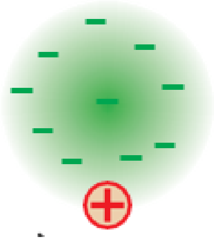
Dipole moment : $\boldsymbol{p} = q\boldsymbol{r}$

Interaction potential energy : $U_I = -\boldsymbol{p} \cdot \boldsymbol{E}$

Induced atomic dipole



$\boldsymbol{p} = \boldsymbol{0}$



$\boldsymbol{p}_{\text{ind}} \neq \boldsymbol{0}$

Quantum operators:

Dipole moment : $\widehat{\boldsymbol{p}} = q\widehat{\boldsymbol{r}}$

Electric field : $\boldsymbol{E}$

Semi-classical interaction

$$\hat{H}_I = -\widehat{\boldsymbol{p}} \cdot \boldsymbol{E}$$

Full or “second” quantized interaction

$$\hat{H}_I = -\widehat{\boldsymbol{p}} \cdot \widehat{\boldsymbol{E}}$$

What kinds of light are `quantum’ ?

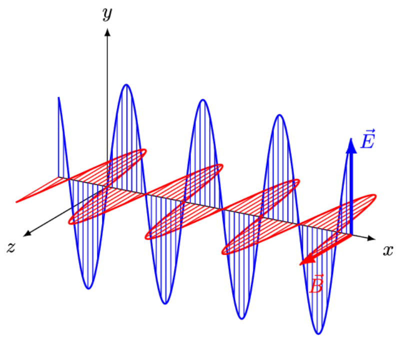
Some kinds of light are more `quantum’ than others

Semi-classical picture :
Atoms treated quantum mechanically/ Field treated classically

$$\boldsymbol{E}_{\text{inc}} = \boldsymbol{e}_{\perp,1}\mathcal{E}\sin(kx - \omega t + \varphi) \qquad \boldsymbol{k} = k\boldsymbol{e}_{\boldsymbol{k}} \qquad \boldsymbol{e}_{\perp,1}\cdot\boldsymbol{e}_{\boldsymbol{k}} = 0$$

$$k^2 = \varepsilon_b\mu_b\epsilon_0\mu_0\omega^2$$

$$\nabla \times \boldsymbol{H} = \frac{\partial \boldsymbol{D}}{\partial t} = \varepsilon_b\epsilon_0\frac{\partial \boldsymbol{E}}{\partial t}$$



$$\boldsymbol{H}_{\text{inc}} = \sqrt{\frac{\varepsilon_b}{\mu_b}\frac{\epsilon_0}{\mu_0}}\,\boldsymbol{e}_{\boldsymbol{k}} \times \boldsymbol{e}_{\perp,1}\mathcal{E}\sin(kx - \omega t + \varphi) = \boldsymbol{e}_{\perp,2}\sqrt{\frac{\varepsilon_b}{\mu_b}\frac{\epsilon_0}{\mu_0}}\,\mathcal{E}\sin(kx - \omega t + \varphi)$$

$$\boldsymbol{\Pi}_{\text{inc}} = \boldsymbol{E}_{\text{inc}} \times \boldsymbol{H}_{\text{inc}} = \boldsymbol{e}_{\boldsymbol{k}}\sqrt{\frac{\varepsilon_b}{\mu_b}\frac{\epsilon_0}{\mu_0}}\,\mathcal{E}^2\sin^2(kx - \omega t + \varphi)$$

$$I_{\boldsymbol{k}} = \langle\boldsymbol{\Pi}_{\text{inc}}\rangle_T \cdot \boldsymbol{e}_{\boldsymbol{k}} = \frac{1}{2}\sqrt{\frac{\varepsilon_b}{\mu_b}\frac{\epsilon_0}{\mu_0}}\,\mathcal{E}^2$$

Intensity is often approximated by $I = |\langle\boldsymbol{\Pi}_{\text{inc}}\rangle_T| = \frac{1}{2}\sqrt{\frac{\varepsilon_b}{\mu_b}\frac{\epsilon_0}{\mu_0}}\,\mathcal{E}^2$

$$I \propto \mathcal{E}^2$$

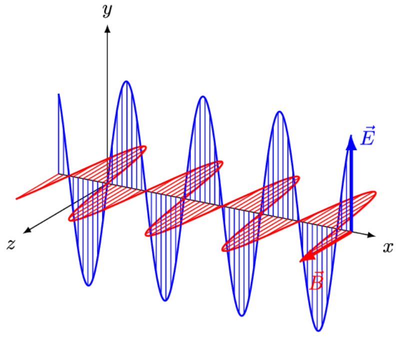
Semi-classical picture :
Time harmonic viewpoint is still possible

$$\boldsymbol{E}_{\text{inc}} = \vec{\boldsymbol{e}}_\perp \mathcal{E} \sin(kx - \omega t + \varphi) = \vec{\boldsymbol{e}}_\perp\ \mathcal{E}\mathrm{Re}\left\{e^{-i\left(\omega t - kx + \varphi - \frac{\pi}{2}\right)}\right\} = \vec{\boldsymbol{e}}_\perp\ \mathrm{Re}\left\{\mathcal{E}_c e^{-i(\omega t - kx)}\right\}$$

$$\mathcal{E}_c = \mathcal{E} e^{i\left(\frac{\pi}{2} - \varphi\right)}$$

$$\nabla \times \boldsymbol{H} = -i\omega \boldsymbol{D} = -i\epsilon_0 \varepsilon_b \omega \boldsymbol{E}$$

$$\nabla \times \boldsymbol{E} = i\omega \boldsymbol{B}$$



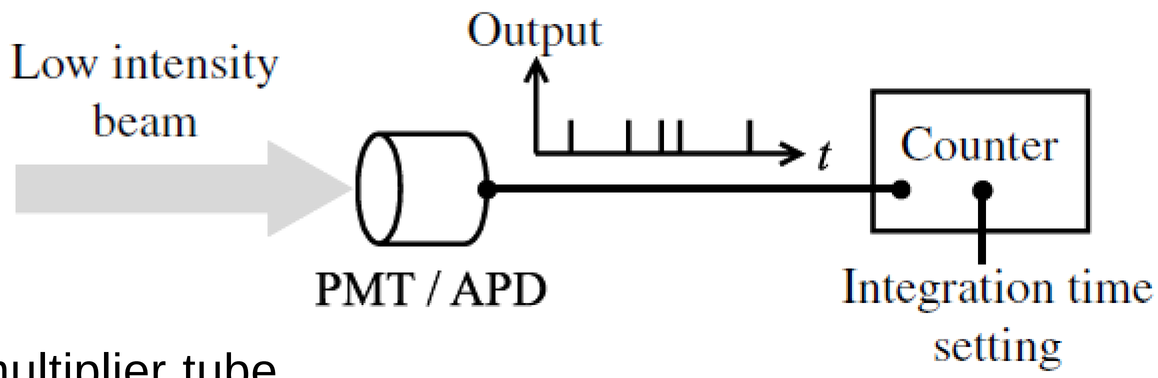
$$\boldsymbol{k} = k\boldsymbol{e}_{\boldsymbol{k}}$$

$$\langle \boldsymbol{\Pi}_{\text{inc}} \rangle_T = \frac{1}{2}\mathrm{Re}\{\boldsymbol{E}^*_{\text{inc}} \times \boldsymbol{H}_{\text{inc}}\} = \frac{1}{2}\sqrt{\frac{\varepsilon_b}{\mu_b}\frac{\epsilon_0}{\mu_0}}\,\|\boldsymbol{E}_{\text{inc}}\|^2 \boldsymbol{e}_\perp = \frac{1}{2}\sqrt{\frac{\varepsilon_b}{\mu_b}\frac{\epsilon_0}{\mu_0}}\,|\mathcal{E}_c|^2 \boldsymbol{e}_\perp = \frac{1}{2}\sqrt{\frac{\varepsilon_b}{\mu_b}\frac{\epsilon_0}{\mu_0}}\,\mathcal{E}^2 \vec{\boldsymbol{e}}_\perp$$

Quantum optics avoids coding phase into complex amplitudes !

$\|\boldsymbol{E}_{\text{inc}}(\boldsymbol{r})\|^2$ **is not** an anology of $|\psi(\boldsymbol{r})|^2$

Intro: photon counting statistics



PMT: Photo multiplier tube

APD: Avalanche photodiode

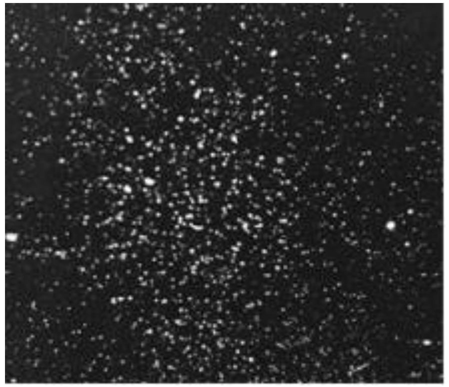
Explanation ?

1. Statistical nature of the photo-detection process ?
2. Intrinsic photon statistics of the light beam ?

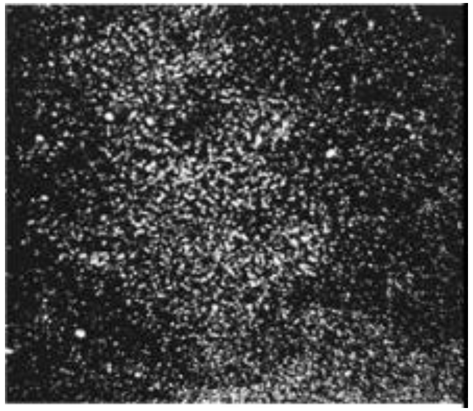
But do photons truly exist ?

(The semi-classical picture of light can explain blackbody radiation, photo-electric effect, stimulated emission (lasers), ultra-fast photography, …)

3×10^3 photons



1.2×10^4 photons



9.3×10^4 photons

2.8×10^7 photons

Photon flux, Φ, quantum efficiency, η, photon count rate , R

$$\Phi = \frac{IA}{\hbar\omega} = \frac{P}{\hbar\omega}\,\text{photons s}^{-1} \qquad\qquad \hbar\omega \cong 2\text{eV}$$

Photon-counting detectors are characterized by their **quantum efficiency:** η $\qquad \eta \sim 0.1$

Number of photons detected in a counting time T: $N(T) = \eta \frac{PT}{\hbar\omega}$

Photon count rate $\mathcal{R} = \frac{N}{T} = \eta\Phi = \eta\frac{P}{\hbar\omega}$.

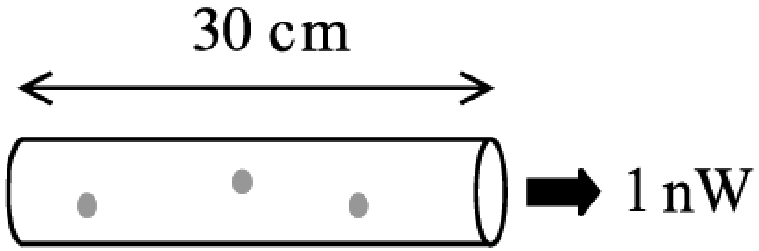
Dead times of typical detectors $\sim 1\mu s = 10^{-6}\text{s} \Longrightarrow \mathcal{R} \lesssim 10^{6} \Longrightarrow P \sim \frac{\mathcal{R}}{\eta}\hbar\omega \sim 2 \times 10^{7} \times 1.6 \times 10^{-19} \lesssim 10^{-12}\text{W} = 1\text{pW}$

Consider a beam of light of photons of energy 2.0 eV with an average of power of 1nW.

Such a beam could be obtained by taking a He:Ne laser operating at 633 nm with a power of 1 mW attenuated by a factor 10^6 with appropriate filters

$$\Phi = \frac{P}{\hbar\omega} = \frac{10^{-9}}{2\times1.6\times10^{-19}} \cong 3.1 \times 10^9 \text{ photons s}^{-1}$$

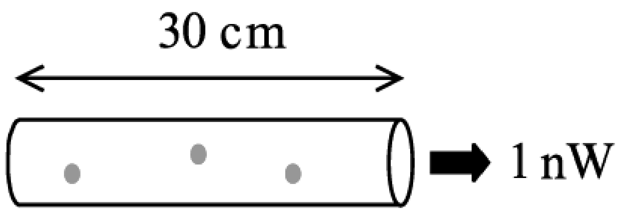
T =1ns $\Longrightarrow L = c\,T \cong 0{,}3\text{m}$ $\Longrightarrow n = T\Phi = \Phi L/c \sim$ 3 photons in 30 cm of beam.



$n =$ 1, 6, 3, 1, 2, 2, 4, 4, 2, 3, 4, 3, 1, 3, 6, 5, 0, 4, 1, 1, 6, 2, 2, 6, 4, 1, 4, 3, 4, 6

$\bar{n} \cong$3.16 $\Delta n \cong$1.81

Photon detection statistics in the semi-classical picture ?



Divide the length L into N segments so small that the probability of detecting more than 1 photon is negligible

The probability of finding n sub-segments containing one photon and $(N - n)$ containing no photons, in any possible order is given by the binomial distribution:

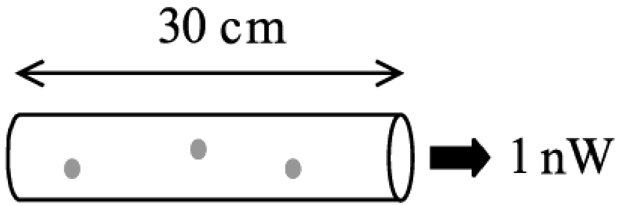
$$\mathcal{P}(n) = \frac{N!}{n!\,(N-n)!} p^n (1-p)^{N-n} \qquad p = \frac{\bar{n}}{N}$$

$$\mathcal{P}(n) = \frac{1}{n!} \left(\frac{N!}{(N-n)!\,N^n} \right) \bar{n}^n \left(1 - \frac{\bar{n}}{N}\right)^{N-n} \qquad \lim_{N\longrightarrow\infty} \left(\frac{N!}{(N-n)!\,N^n} \right) \longrightarrow 1$$

Stirling's Formula $\lim_{N\longrightarrow\infty} \ln(N!) = N\ln(N) - N$

Photon detection statistics in the semi-classical picture

$$\mathcal{P}(n) = \frac{1}{n!}\left(\frac{N!}{(N-n)!\,N^n}\right)\bar{n}^n\left(1-\frac{\bar{n}}{N}\right)^{N-n} \qquad \lim_{N\to\infty}\left(\frac{N!}{(N-n)!\,N^n}\right)\longrightarrow 1$$

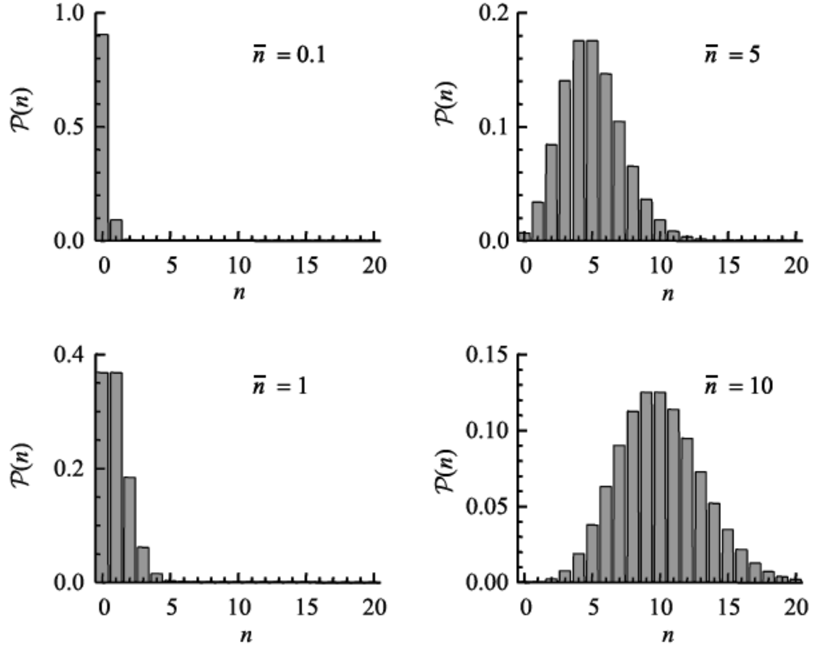


$$\left(1-\frac{\bar{n}}{N}\right)^{N-n} = 1-(N-n)\frac{\bar{n}}{N}+\frac{(N-n)(N-n-1)}{2!}\left(\frac{\bar{n}}{N}\right)^2-\cdots$$

$$\longrightarrow 1-\bar{n}+\frac{1}{2!}(\bar{n})^2+\cdots\longrightarrow \exp(-\bar{n})$$

$$\lim_{N\to\infty}\mathcal{P}(n) = \frac{\bar{n}^n}{n!}e^{-\bar{n}}$$

Poisson distribution



Statistical variations of the photon distribution is **Poissonian**

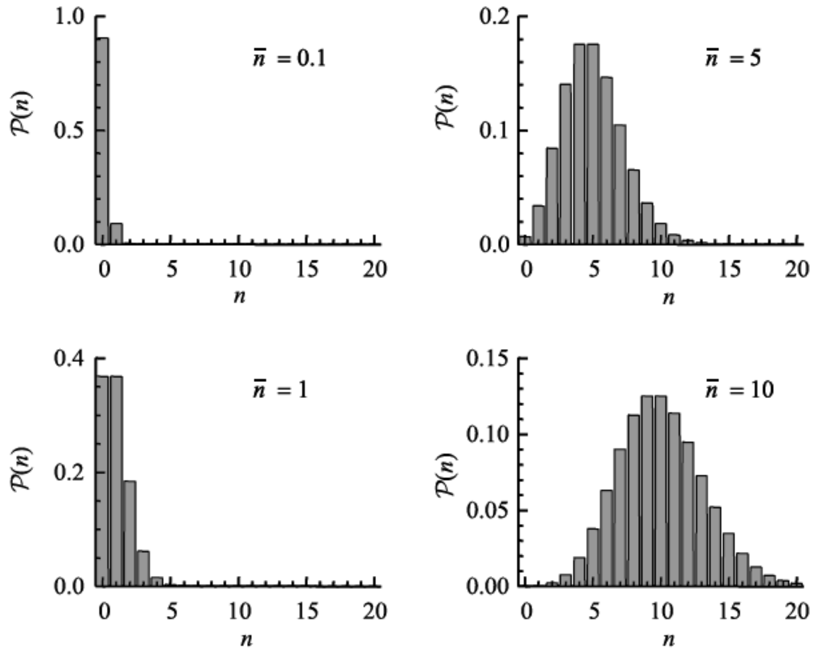
$$\lim_{N\longrightarrow\infty}\mathcal{P}(n)=\frac{\bar{n}^n}{n!}e^{-\bar{n}}$$

$$\bar{n}=\sum_{n=0}^{\infty}n\mathcal{P}(n)$$

$$(\Delta n)^2=\sum_{n=0}^{\infty}(n-\bar{n})^2\mathcal{P}(n)=\bar{n}$$

$$\Delta n=\sqrt{\bar{n}}$$

$$\frac{\Delta n}{\bar{n}}=\frac{1}{\sqrt{\bar{n}}}$$



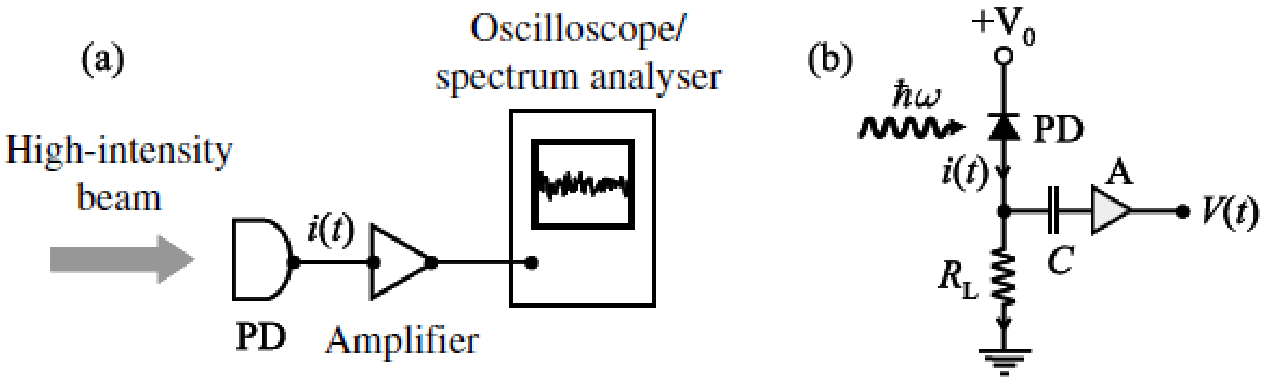
For large $\bar{n}$

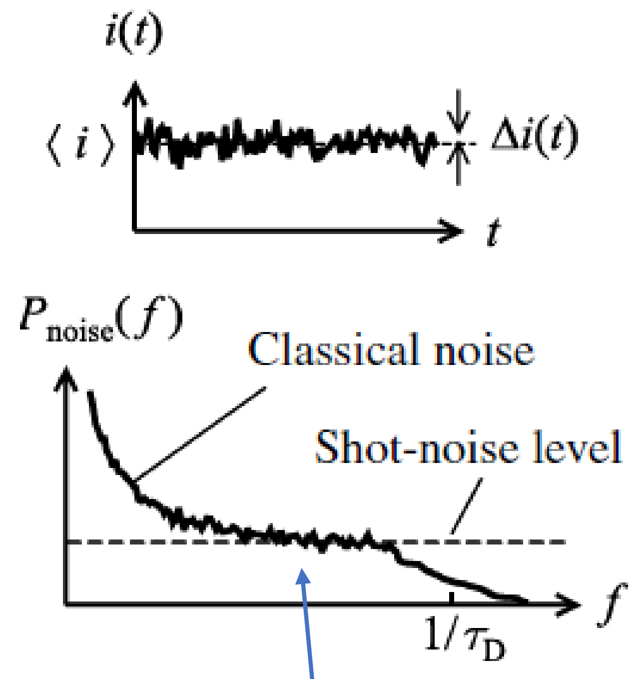
$$P(n)\cong\frac{1}{\sqrt{2\pi\bar{n}}}e^{-\frac{1}{2}\frac{(n-\bar{n})^2}{\bar{n}}}$$

Gaussian distribution

Statistical variations of the photo current

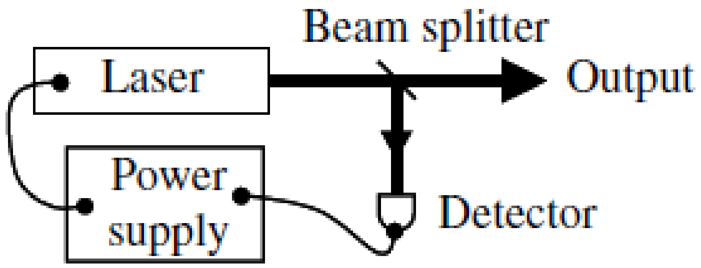
$\Delta n = \sqrt{\langle n \rangle} \rightarrow \Delta i = \sqrt{\langle i \rangle}$

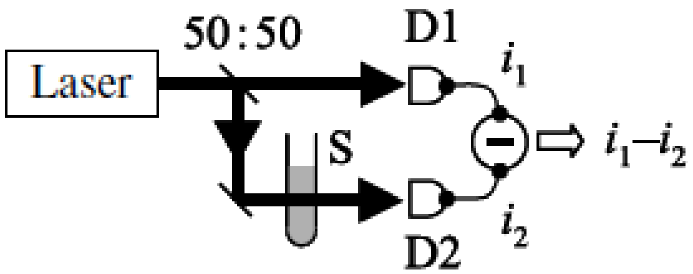




Standard “quantum limit”

Classical noise can be reduced by electronics





Quantum noise cannot !

Going beyond the standard quantum limit requires “quantum” light ! Notably $\Delta n < \sqrt{\langle n \rangle}$

Responsivity of a photodetector

- Photodiodes are semiconductor devices commonly used for measuring higher power light which generate a current in an external circuit proportional to Φ.

- For an incident photon flux Φ, the **photocurrent** current i, is given by: $i = \eta e \Phi = \frac{\eta e}{\hbar\omega}$

- The ratio $\frac{i}{P} = \frac{\eta e}{\hbar\omega}$ is called the **responsivity** of the photodiode and has the units of $\mathrm{A.W^{-1}}$
  Values are given by manufacturers.

- One can use the **responsivity** to deduce the quantum efficiency of a photodetector

$$\eta = \frac{\hbar\omega}{e}\frac{i}{P}$$

Photo-detectors are never perfect !

Photo count number N :

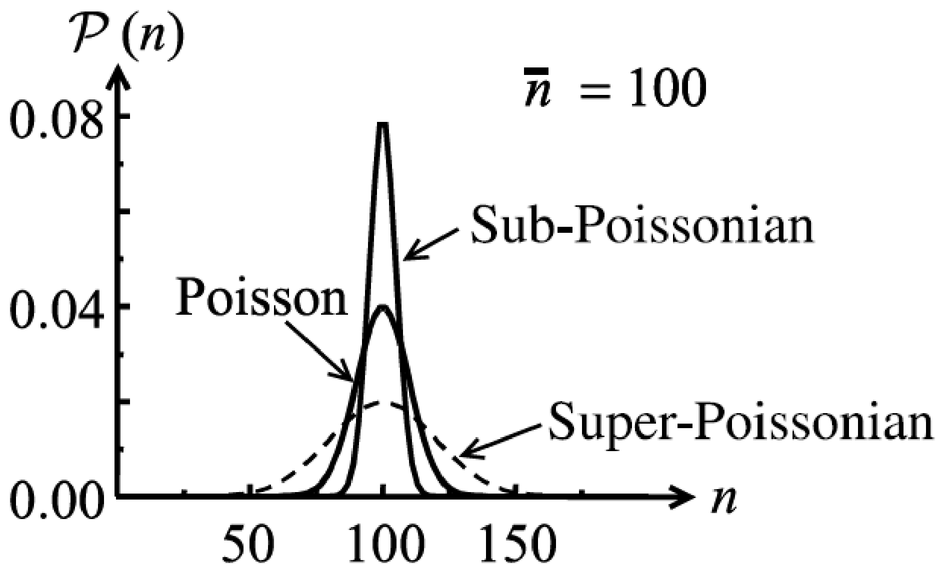
Quantum efficiency : $\eta \equiv \frac{\langle N \rangle}{\langle n \rangle} = \frac{\bar{N}}{\bar{n}}$

Ordinary statistical analysis : $\langle \Delta N \rangle^2 = \eta^2 \langle \Delta n \rangle^2 + \eta(1-\eta)\bar{n}$

Poissonian statistics : $(\Delta n)^2 = \bar{n}$

- $\eta = 1$: $\langle \Delta N \rangle = \langle \Delta n \rangle$
- Poissonian distributions (lasers) $\forall \eta$: $\langle \Delta n \rangle^2 = \langle n \rangle = \bar{n} \Longrightarrow \langle \Delta N \rangle^2 = \eta \bar{n} \Longrightarrow \langle \Delta N \rangle^2 = \eta \bar{n} = \bar{N}$
- $\eta \ll 1$: $\langle \Delta N \rangle^2 = \eta \bar{n} = \bar{N}$, meaning that count rate statistics are Poissonian

Photon statistics are one signature of "Quantum" (i.e non-classical) light



- Laser light has Poissonian statistics

- Classical (non-laser) light is characterized by super-Poissonian distributions
  (can be described by either classical or quantum statistical ensembles)

- Sub-Poissonian statistics **can only** be achieved **with "Quantum" light!**
  (Notably however some kinds of quantum light may also have super-Poissonian statistics!)

A few take away messages

- High quality laser light has Poissonian statistics (Poissonian statistics is a characteristic of random processes involving integer numbers – notably radio-activity)
- Sub-Poissonian statistics is a signature of Quantum light.
- Photo-detectors need to have high quantum efficiencies in order to detect sub-Poissonian statistics
- Low efficiency detectors tend towards Poissonian statistics even for sub-Poissonian light
- Super-Poissonian statistics does not necessarily imply "Classical light" (some forms of quantum light have super-Poissonian statistics)

Single mode radiation

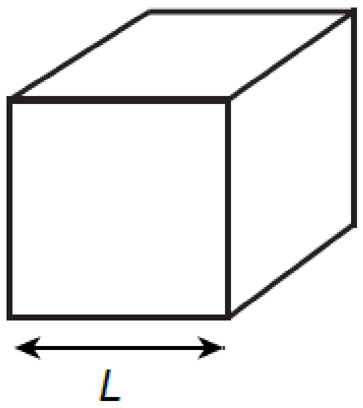
Discrete modal solutions of the wave Equation

In infinite space, $\boldsymbol{k}$ is continuously distributed

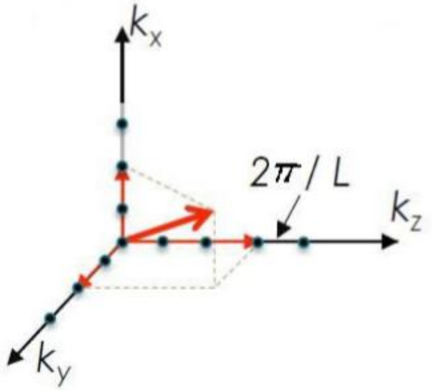
For quantization we prefers to work in terms discrete modes, ℓ , which are discrete and discontinuously distributed **(periodic boundary conditions)**

$$k_x = \frac{2\pi}{L} n_x \qquad k_y = \frac{2\pi}{L} n_y \qquad k_z = \frac{2\pi}{L} n_z$$

$n_i \in -\infty$,…,-1,0,1,2,…∞



$$\boldsymbol{k}_\ell = (k_x, k_y, k_z)$$



$$\boldsymbol{A}(\boldsymbol{r}, t) = \sum_\ell \boldsymbol{\epsilon}_\ell \left(A_\ell(t) e^{i\boldsymbol{k}_\ell \cdot \boldsymbol{r}} + A_\ell^*(t) e^{-i\boldsymbol{k}_\ell \cdot \boldsymbol{r}} \right)$$

$$\boldsymbol{E}(\boldsymbol{r}, t) = -\frac{\partial \boldsymbol{A}(\boldsymbol{r}, t)}{\partial t} = -\sum_\ell \boldsymbol{\epsilon}_\ell \left(\dot{A}_\ell(t) e^{i\boldsymbol{k}_\ell \cdot \boldsymbol{r}} + \dot{A}_\ell^*(t) e^{-i\boldsymbol{k}_\ell \cdot \boldsymbol{r}} \right) = \sum_\ell \boldsymbol{\epsilon}_\ell \left(E_\ell(t) e^{i\boldsymbol{k}_\ell \cdot \boldsymbol{r}} + E_\ell^*(t) e^{-i\boldsymbol{k}_\ell \cdot \boldsymbol{r}} \right)$$

$$\boldsymbol{B}(\boldsymbol{r}, t) = \nabla \times \boldsymbol{A}(\boldsymbol{r}, t) = \sum_\ell i(\boldsymbol{k}_\ell \times \boldsymbol{\epsilon}_\ell) \left(A_\ell(t) e^{i\boldsymbol{k}_\ell \cdot \boldsymbol{r}} - A_\ell^*(t) e^{-i\boldsymbol{k}_\ell \cdot \boldsymbol{r}} \right)$$

Maxwell equations reduced to equations on the coefficients

$$\boldsymbol{E}(\boldsymbol{r},t) = -\frac{\partial \boldsymbol{A}(\boldsymbol{r},t)}{\partial t} = -\sum_{\ell} \boldsymbol{\epsilon}_{\ell}\left(\dot{A}_{\ell}(t)e^{i\boldsymbol{k}_{\ell}\cdot\boldsymbol{r}} + \dot{A}_{\ell}^{*}(t)e^{-i\boldsymbol{k}_{\ell}\cdot\boldsymbol{r}}\right) = \sum_{\ell} \boldsymbol{\epsilon}_{\ell}\left(E_{\ell}(t)e^{i\boldsymbol{k}_{\ell}\cdot\boldsymbol{r}} + E_{\ell}^{*}(t)e^{-i\boldsymbol{k}_{\ell}\cdot\boldsymbol{r}}\right)$$

$$\mathbf{1)} \quad \dot{A}_{\ell}(t) = -E_{\ell}(t)$$

Since $\boldsymbol{B} = \nabla \times \boldsymbol{A}$, the time evolution of $\boldsymbol{B}(\boldsymbol{r},t)$ is now determined

Time evolution of $\boldsymbol{E}(\boldsymbol{r},t)$ can be determined from: $\frac{\partial \boldsymbol{E}}{\partial t} = c^2 \nabla \times \boldsymbol{B}$

$$\omega_{\ell} \equiv c k_{\ell}$$

$$\sum_{\ell} \boldsymbol{\epsilon}_{\ell}\left(\dot{E}_{\ell}(t)e^{i\boldsymbol{k}_{\ell}\cdot\boldsymbol{r}} + \dot{E}_{\ell}^{*}(t)e^{-i\boldsymbol{k}_{\ell}\cdot\boldsymbol{r}}\right) = c^2 \nabla \times \nabla \times \boldsymbol{A}(\boldsymbol{r},t) = -c^2 \Delta \boldsymbol{A}(\boldsymbol{r},t) = \sum_{\ell} \omega_{\ell}^{2}\left(A_{\ell}(t)e^{i\boldsymbol{k}_{\ell}\cdot\boldsymbol{r}} + A_{\ell}(t)e^{-i\boldsymbol{k}_{\ell}\cdot\boldsymbol{r}}\right)$$

$$\mathbf{2)} \quad \dot{E}_{\ell}(t) = \omega_{\ell}^{2} A_{\ell}(t)$$

Normal mode formulation of the Maxwell equations

`Normal mode' amplitude $\alpha_\ell(t)$

1-photon electric field "amplitude"

$$\alpha_\ell(t) \equiv \frac{1}{2\mathcal{E}^{(1)}_{\text{p.w.}}}[\omega_\ell A_\ell(t) - iE_\ell(t)] \equiv \frac{1}{\sqrt{2\hbar}}[Q_\ell(t) - iE_\ell(t)]$$

$$\mathcal{E}^{(1)}_{\text{p.w.}} \equiv \sqrt{\frac{\hbar\omega_\ell}{2\epsilon_0 V}}$$

1) $\dot{A}_\ell(t) = -E_\ell(t)$ 2) $\dot{E}_\ell(t) = \omega_\ell^2 A_\ell(t)$

Time evolution (Normal mode) $\dot{\alpha}_\ell(t) = -i\omega_\ell[\omega_\ell A_\ell(t) - iE_\ell(t)] = -i\omega_\ell\alpha_\ell(t)$

Normal mode equation contains all the information on the temporal evolution of the radiative EM field:

$$\dot{\alpha}_\ell(t) = -i\omega_\ell\alpha_\ell(t)$$

Normal mode formulation of the Maxwell equations

Canonically conjugate pairs $Q_\ell(t)$ and $P_\ell(t)$:

$$\alpha_\ell(t) \equiv \frac{1}{\mathcal{E}_{\text{p.w.}}^{(1)}}[\omega_\ell A_\ell(t) - iE_\ell(t)] \equiv \frac{1}{\sqrt{2\hbar}}[Q_\ell(t) + iP_\ell(t)]$$

$$Q_\ell(t) = \sqrt{\frac{\hbar}{2}}\frac{\omega_\ell A_\ell(t)}{\mathcal{E}_{\text{p.w.}}^{(1)}} = \sqrt{\frac{\hbar}{2}}[\alpha_\ell(t) + i\alpha_\ell^*(t)]$$

$$P_\ell(t) = -\sqrt{\frac{\hbar}{2}}\frac{E_\ell(t)}{\mathcal{E}_{\text{p.w.}}^{(1)}} = \frac{1}{i}\sqrt{\frac{\hbar}{2}}[\alpha_\ell(t) - \alpha_\ell^*(t)]$$

Given $\dot{A}_\ell(t) = -E_\ell(t)$ and $\dot{E}_\ell(t) = \omega_\ell^2 A_\ell(t)$

⇓

$$\dot{Q}_\ell(t) = \omega_\ell\, P_\ell(t) \qquad \dot{P}_\ell(t) = -\omega_\ell\, Q_\ell(t)$$

These equations are also equivalent to the EM field equations

Hamiltonian of the electromagnetic field

$$H = \sum_{\ell} H_{\ell} \qquad H_{\ell} = \frac{\omega_{\ell}}{2}\left(Q_{\ell}^2 + P_{\ell}^2\right)$$

Hamilton equations. of a harmonic oscillator

$$\frac{dQ_{\ell}(t)}{dt} = \frac{\partial H}{\partial P_{\ell}} = \omega_{\ell}\, P_{\ell}(t)$$

$$\frac{dP_{\ell}(t)}{dt} = -\frac{\partial H}{\partial Q_{\ell}} = -\omega_{\ell} Q_{\ell}(t)$$

$$\alpha_{\ell}(t) \equiv \frac{1}{\sqrt{2}}\left(Q_{\ell}(t) + iP_{\ell}(t)\right)$$

$$Q_{\ell}(t) = \frac{1}{\sqrt{2}}[\alpha_{\ell}(t) + \alpha_{\ell}^{*}(t)] \qquad P_{\ell}(t) = \frac{1}{i\sqrt{2}}[\alpha_{\ell}(t) - \alpha_{\ell}^{*}(t)]$$

Fields in terms of the complex-valued normal mode amplitude:

$$\alpha_\ell(t) \equiv \frac{1}{\mathcal{E}^{(1)}_{\text{p.w.}}}[\omega_\ell A_\ell(t) - iE_\ell(t)] \equiv \frac{1}{\sqrt{2\hbar}}[Q_\ell(t) + iP_\ell(t)]$$

Since :
$$\sqrt{\frac{\hbar}{2}}\frac{\omega_\ell A_\ell(t)}{\mathcal{E}^{(1)}_{\text{p.w.}}} = Q_\ell(t) = \sqrt{\frac{\hbar}{2}}[\alpha_\ell(t) + \alpha^*_\ell(t)]$$

$$\boldsymbol{A}(\boldsymbol{r},t) = \sum_\ell \boldsymbol{\epsilon}_\ell\left(A_\ell(t)e^{i\boldsymbol{k}_\ell\cdot\boldsymbol{r}} + A^*_\ell(t)e^{-i\boldsymbol{k}_\ell\cdot\boldsymbol{r}}\right) = \sum_\ell \frac{\boldsymbol{\epsilon}_\ell\mathcal{E}^{(1)}_{\text{p.w.}}}{\omega_\ell}\left[\alpha_\ell(t)e^{i\boldsymbol{k}_\ell\cdot\boldsymbol{r}} + \text{c.c.}\right]$$
$$= \sum_\ell \frac{\boldsymbol{\epsilon}_\ell\mathcal{E}^{(1)}_{\text{p.w.}}}{\omega_\ell}\alpha_\ell(t)e^{i\boldsymbol{k}_\ell\cdot\boldsymbol{r}} + \text{c.c.}$$

$$\boldsymbol{B}(\boldsymbol{r},t) = \nabla\times\boldsymbol{A}(\boldsymbol{r},t) = \sum_\ell \mathcal{E}^{(1)}_{\text{p.w.}}\frac{i\boldsymbol{k}_\ell\times\boldsymbol{\epsilon}_\ell}{\omega_\ell}\left[\alpha_\ell(t)e^{i\boldsymbol{k}_\ell\cdot\boldsymbol{r}} - \alpha^*_\ell(t)e^{-i\boldsymbol{k}_\ell\cdot\boldsymbol{r}}\right]$$
$$= \sum_\ell \frac{\boldsymbol{k}_\ell\times\boldsymbol{\epsilon}_\ell}{\omega_\ell}\mathcal{E}^{(1)}_{\text{p.w.}}\left(i\alpha_\ell(t)e^{i\boldsymbol{k}_\ell\cdot\boldsymbol{r}} + \text{c.c.}\right)$$

Fields in terms of the complex-valued normal mode amplitude:

Since : $P_\ell(t) = -\sqrt{\frac{\hbar}{2}}\frac{E_\ell(t)}{\mathcal{E}^{(1)}_{\text{p.w.}}} = \frac{1}{i}\sqrt{\frac{\hbar}{2}}[\alpha_\ell(t) - \alpha^*_\ell(t)]$ ⟹ $E_\ell(t) = i\mathcal{E}^{(1)}_{\text{p.w.}}[\alpha_\ell(t) - \alpha^*_\ell(t)]$

$$\boldsymbol{E}(\boldsymbol{r},t) = \sum_\ell \boldsymbol{\epsilon}_\ell\left(E_\ell(t)e^{i\boldsymbol{k}_\ell\cdot\boldsymbol{r}} + E^*_\ell(t)e^{-i\boldsymbol{k}_\ell\cdot\boldsymbol{r}}\right) = \sum_\ell \boldsymbol{\epsilon}_\ell\mathcal{E}^{(1)}_{\text{p.w.}}\left[i\alpha_\ell(t)e^{i\boldsymbol{k}_\ell\cdot\boldsymbol{r}} - i\alpha^*_\ell(t)e^{-i\boldsymbol{k}_\ell\cdot\boldsymbol{r}}\right]$$

$$= \sum_\ell \left[i\boldsymbol{\epsilon}_\ell\mathcal{E}^{(1)}_{\text{p.w.}}\alpha_\ell(t)e^{i\boldsymbol{k}_\ell\cdot\boldsymbol{r}} + \text{c. c.}\right]$$

The 1-photon amplitude, $\mathcal{E}^{(1)}_{\text{p.w.}}(\omega_\ell)$ is determined as the value which expresses the classical energy, $H = \frac{\epsilon_0}{2}\int d^3r\,(\boldsymbol{E}^2 + c^2\boldsymbol{B}^2)$, as a sum of harmonic oscillators

$$\boldsymbol{E}(\boldsymbol{r},t) = \sum_\ell i\boldsymbol{\epsilon}_\ell \mathcal{E}^{(1)}_{\text{p.w.}}\left[i\alpha_\ell(t)e^{i\boldsymbol{k}_\ell\cdot\boldsymbol{r}} + \text{c.c.}\right]$$

$$\boldsymbol{B}(\boldsymbol{r},t) = \sum_\ell \frac{\boldsymbol{k}_\ell \times \boldsymbol{\epsilon}_\ell}{\omega_\ell}\mathcal{E}^{(1)}_{\text{p.w.}}\left(i\alpha_\ell(t)e^{i\boldsymbol{k}_\ell\cdot\boldsymbol{r}} + \text{c.c.}\right)$$

$$\int d^3r\boldsymbol{E}^2 = V\sum_\ell \left[\mathcal{E}^{(1)}_{\text{p.w.}}\right]^2 \{[\alpha_\ell(t)\alpha^*_\ell(t) + \alpha_\ell(t)\alpha_{-\ell}(t)] + c.c.\}$$

$$\int d^3rc^2\boldsymbol{B}^2 = V\sum_\ell \left[\mathcal{E}^{(1)}_{\text{p.w.}}\right]^2 \{[\alpha_\ell(t)\alpha^*_\ell(t) - \alpha_\ell(t)\alpha_{-\ell}(t)] + c.c.\}$$

$$\mathcal{E}^{(1)}_{\text{p.w.}} \equiv \sqrt{\frac{\hbar\omega_\ell}{2\epsilon_0 V}}$$

$$H = \frac{\epsilon_0}{2}\int d^3r(\boldsymbol{E}^2 + c^2\boldsymbol{B}^2) = \sum_\ell \frac{\hbar\omega_\ell}{2}[\alpha_\ell(t)\alpha^*_\ell(t) + \alpha^*_\ell(t)\alpha_\ell(t)] \equiv \sum_\ell H_\ell$$

Resume of modal field expansions:

$$\boldsymbol{A}(\boldsymbol{r},t) = \sum_{\ell} \left[\frac{\boldsymbol{\epsilon}_{\ell} \mathcal{E}_{\text{p.w.}}^{(1)}}{\omega_{\ell}} \alpha_{\ell}(t) e^{i\boldsymbol{k}_{\ell}\cdot\boldsymbol{r}} + \text{c.c.} \right]$$

$$\boldsymbol{E}(\boldsymbol{r},t) = \sum_{\ell} \left[i\boldsymbol{\epsilon}_{\ell} \mathcal{E}_{\text{p.w.}}^{(1)} \alpha_{\ell}(t) e^{i\boldsymbol{k}_{\ell}\cdot\boldsymbol{r}} + \text{c.c.} \right]$$

$$\boldsymbol{B}(\boldsymbol{r},t) = \sum_{\ell} \left[i \frac{\boldsymbol{k}_{\ell} \times \boldsymbol{\epsilon}_{\ell}}{\omega_{\ell}} \mathcal{E}_{\text{p.w.}}^{(1)} \alpha_{\ell}(t) e^{i\boldsymbol{k}_{\ell}\cdot\boldsymbol{r}} + \text{c.c.} \right]$$

1-photon electric field “amplitude”

$$\mathcal{E}_{\text{p.w.}}^{(1)}(\omega_{\ell}) \equiv \sqrt{\frac{\hbar\omega_{\ell}}{2\epsilon_0 V}}$$

This convention for modal field development corresponds to that used by Glauber in his pioneering papers of 1963 and remains by far the most common form.

This convention will be used in the rest of this course and in the homework.

Quantum field (Heisenberg picture) : `second quantization'

$$\hat{\boldsymbol{E}}_{\mathrm{H}}(\boldsymbol{r},t) = \sum_{\ell} \mathcal{E}^{(1)}_{\mathrm{p.w.}}\left[i\boldsymbol{\epsilon}_{\ell}\hat{a}_{\ell}e^{-i(\omega_{\ell}t-\boldsymbol{k}_{\ell}\cdot\boldsymbol{r})} - i\bar{\boldsymbol{\epsilon}}_{\ell}\hat{a}^{\dagger}_{\ell}e^{i(\omega_{\ell}t-\boldsymbol{k}_{\ell}\cdot\boldsymbol{r})}\right]$$

$$\mathcal{E}^{(1)}_{\mathrm{p.w.}} \equiv \sqrt{\frac{\hbar\omega_{\ell}}{2\epsilon_0 V}}$$

$$= \sum_{\ell} \mathcal{E}^{(1)}_{\mathrm{p.w.}}\left[\boldsymbol{\epsilon}_{\ell}\hat{a}_{\ell}e^{-i\left(\omega_{\ell}t-\boldsymbol{k}_{\ell}\cdot\boldsymbol{r}-\frac{\pi}{2}\right)} + \bar{\boldsymbol{\epsilon}}_{\ell}\hat{a}^{\dagger}_{\ell}e^{i\left(\omega_{\ell}t-\boldsymbol{k}_{\ell}\cdot\boldsymbol{r}-\frac{\pi}{2}\right)}\right]$$

$$\hat{\boldsymbol{E}}_{\mathrm{H}}(\boldsymbol{r},t) = \sum_{\ell} \boldsymbol{\epsilon}_{\ell}\mathcal{E}^{(1)}_{\mathrm{p.w.}}\left[\hat{a}_{\ell}e^{-i\chi_{\ell}} + \hat{a}^{\dagger}_{\ell}e^{i\chi_{\ell}}\right] = \hat{\boldsymbol{E}}^{(+)}(\chi_{\ell})+\hat{\boldsymbol{E}}^{(-)}(\chi_{\ell})$$

$$\chi_{\ell} \equiv \omega_{\ell}t - \boldsymbol{k}_{\ell}\cdot\boldsymbol{r} - \frac{\pi}{2}$$

Quantum field (Heisenberg picture) : `second quantization’

$$\mathcal{E}^{(1)}_{\text{p.w.}} \equiv \sqrt{\frac{\hbar\omega_\ell}{2\epsilon_0 V}}$$

$$\hat{\boldsymbol{E}}_{\text{H}}(\boldsymbol{r},t) = \sum_\ell \mathcal{E}^{(1)}_{\text{p.w.}}\left[\boldsymbol{\epsilon}_\ell \hat{a}_\ell e^{-i\chi_\ell} + \bar{\boldsymbol{\epsilon}}_\ell \hat{a}^\dagger_\ell e^{i\chi_\ell}\right] = \hat{\boldsymbol{E}}^{(+)}(\chi_\ell) + \hat{\boldsymbol{E}}^{(-)}(\chi_\ell)$$

$$\hat{\boldsymbol{B}}_{\text{H}}(\boldsymbol{r},t) = \sum_\ell \frac{\mathcal{E}^{(1)}_{\text{p.w.}}}{c}\left[\boldsymbol{k}_\ell \times \boldsymbol{\epsilon}_\ell \hat{a}_\ell e^{-i\chi_\ell} + \boldsymbol{k}_\ell \times \bar{\boldsymbol{\epsilon}}_\ell \hat{a}^\dagger_\ell e^{i\chi_\ell}\right]$$

$$\chi_\ell \equiv \omega_\ell t - \boldsymbol{k}_\ell \cdot \boldsymbol{r} - \frac{\pi}{2}$$

Second quantization means that field amplitude are now operators

Commutation relations for the ladder operators $\left[\hat{a}_\ell, \hat{a}_k^\dagger\right] = \delta_{\ell,k}$

$[\hat{a}_\ell^\dagger, \hat{a}_k^\dagger] = 0 \qquad [\hat{a}_\ell, \hat{a}_k] = 0$

The ladder operators, $\hat{a}_\ell^\dagger$ and $\hat{a}_\ell$ respectively create and destroy photons in a mode ℓ :

$$\hat{a}_\ell^\dagger |n\rangle_\ell = \sqrt{n+1}|n+1\rangle_\ell$$
$$\hat{a}_\ell |n\rangle_\ell = \sqrt{n}|n-1\rangle_\ell$$

The number operator, $\hat{N}_\ell = \hat{a}_\ell^\dagger \hat{a}_\ell$ counts the number of excitations, i.e. ‘photons’ in mode ℓ :

$$\hat{N}_\ell |n\rangle_\ell = \hat{a}_\ell^\dagger \hat{a}_\ell |n\rangle_\ell = n|n\rangle_\ell$$

Single-mode field operator : $\hat{\boldsymbol{E}}_\ell(\chi_\ell)$

$$\hat{\boldsymbol{E}}_\ell(\chi_\ell) = \boldsymbol{\epsilon}_\ell \mathcal{E}^{(1)}_{\text{p.w.}}\left[\hat{a}_\ell e^{-i\chi_\ell} + \hat{a}^\dagger_\ell e^{i\chi_\ell}\right] = \hat{\boldsymbol{E}}_\ell{}^{(+)}(\chi_\ell) + \hat{\boldsymbol{E}}_\ell{}^{(-)}(\chi_\ell) \qquad \chi_\ell \equiv \omega_\ell t - \boldsymbol{k}_\ell \cdot \boldsymbol{r} - \frac{\pi}{2}$$

Field operator $\hat{\boldsymbol{E}}_\ell$ in "natural" units : $\mathcal{E}^{(1)}_{\text{p.w.}} = \sqrt{\frac{\hbar\omega_\ell}{2\epsilon_0 V}} \to 1$

Drop the ℓ index and $\hat{\boldsymbol{E}}(\chi) = \boldsymbol{\epsilon}\hat{E}(\chi)$

$$\hat{E}(\chi) = \hat{a}e^{-i\chi} + \hat{a}^\dagger e^{i\chi} = \hat{E}^{(+)}(\chi) + \hat{E}^{(-)}(\chi)$$

$$\chi = \omega t - \boldsymbol{k} \cdot \boldsymbol{r} - \frac{\pi}{2}$$

Single Radiation Mode - Fock States

Fock state (number state) : $|n_\ell\rangle \rightarrow |n\rangle$ n `photons' in radiation mode ℓ

Eigenstate of number operator : $\hat{N}_\ell \rightarrow \hat{N} = \hat{n}$
and the normal ordered Hamiltonian of the mode : $\hat{H}_\ell : \rightarrow : \hat{H} := \hbar\omega\left(\hat{a}^\dagger \hat{a}\right) = \hbar\omega\hat{N}$

$$:\hat{H}:|n\rangle = \hbar\omega\left(\hat{a}^\dagger \hat{a}\right)|n\rangle$$

$$= \hbar\omega\hat{N}|n\rangle$$

$$= \hbar\omega n|n\rangle$$

$$= E_n|n\rangle = \hbar\omega_n|n\rangle$$

$\omega_n = n\omega$ $\omega = \omega_\ell$ is the angular frequency of the mode ℓ

Expectation value of the number of photons

$$\langle\hat{N}\rangle \equiv \langle n|\,\hat{N}\,|n\rangle = \langle n|\,n\,|n\rangle = n$$

Average value and fluctuations of observables

Average value of an observable: $\langle O \rangle = \langle O \rangle_\psi \equiv \langle \psi | \hat{O} | \psi \rangle$

Since $\hat{O}$ is Hermitian, $\langle O \rangle_\psi$ is real valued

Fluctuations of measurement of the observable, O :

$$(\Delta O)^2 \equiv \langle \psi | \left(\hat{O} - \langle \hat{O} \rangle \right)^2 | \psi \rangle$$

$$= \langle \psi | \hat{O}^2 - 2\hat{O}\langle \hat{O} \rangle + \langle \hat{O} \rangle^2 | \psi \rangle$$

$$= \langle \psi | \hat{O}^2 | \psi \rangle - 2\langle \hat{O} \rangle \langle \psi | \hat{O} | \psi \rangle + \langle \hat{O} \rangle^2 \langle \psi | \psi \rangle$$

$$= \langle \psi | \hat{O}^2 | \psi \rangle - \langle \hat{O} \rangle^2$$

$$= \langle \hat{O}^2 \rangle - \langle \hat{O} \rangle^2$$

Single Radiation Mode - Fock States

Fluctuations in n : $(\Delta n)^2 \equiv \langle n| \left(\hat{N} - \langle\hat{N}\rangle\right)^2 |n\rangle$ $\hat{N} = \hat{a}^\dagger \hat{a}$

$$(\Delta n)^2 = \langle n| \left(\hat{N} - \langle\hat{N}\rangle\right)^2 |n\rangle$$

$$= \langle n| \hat{N}^2 - 2\hat{N}\langle\hat{N}\rangle + \langle\hat{N}\rangle^2 |n\rangle$$

$$= \langle n| \hat{N}^2 |n\rangle - \langle\hat{N}\rangle^2$$

$$= n^2 - n^2 = 0$$

Single Radiation Mode - Fock States

Field expectation value : $\langle E(\chi)\rangle$

$$\langle E(\chi)\rangle = \langle n|\, \hat{a}e^{-i\chi} + \hat{a}^{\dagger}e^{i\chi}|n\rangle$$

$$= e^{-i\chi}\langle n|\hat{a}|n\rangle + e^{i\chi}\langle n|\, \hat{a}^{\dagger}|n\rangle$$

$$= 0$$

Single Radiation Mode - Fock States

Field fluctuations : $(\Delta E(\chi))^2$

$$(\Delta E(\chi))^2 = \langle \hat{E}^2(\chi)\rangle - \langle \hat{E}(\chi)\rangle^2 = \langle n|\left(\hat{a}e^{-i\chi} + \hat{a}^\dagger e^{i\chi}\right)^2|n\rangle \quad (\mathrm{n.u})$$

$$\langle E(\chi)\rangle = 0$$

$$= \langle n|\, \hat{a}\hat{a}e^{-2i\chi} + \hat{a}^\dagger\hat{a}^\dagger e^{2i\chi} + \hat{a}\hat{a}^\dagger + \hat{a}^\dagger\hat{a}|n\rangle$$

$$\hat{a}\hat{a}^\dagger = 1 + \hat{a}^\dagger\hat{a}$$

$$= \langle n|\, 2\hat{a}^\dagger\hat{a} + 1|n\rangle$$

$$= \langle n|\, 2\hat{N} + 1|n\rangle$$

$$= 2n + 1$$

$$\Delta E(\chi) = \sqrt{2n+1}(\mathrm{n.u}) = \sqrt{2n+1}\,\mathcal{E}^{(1)}_{\mathrm{p.w.}}$$

$$\langle 0|\, \hat{E}^2(\chi)|0\rangle = 1(\mathrm{n.u}) = \mathcal{E}^{(1)}_{\mathrm{p.w.}}$$

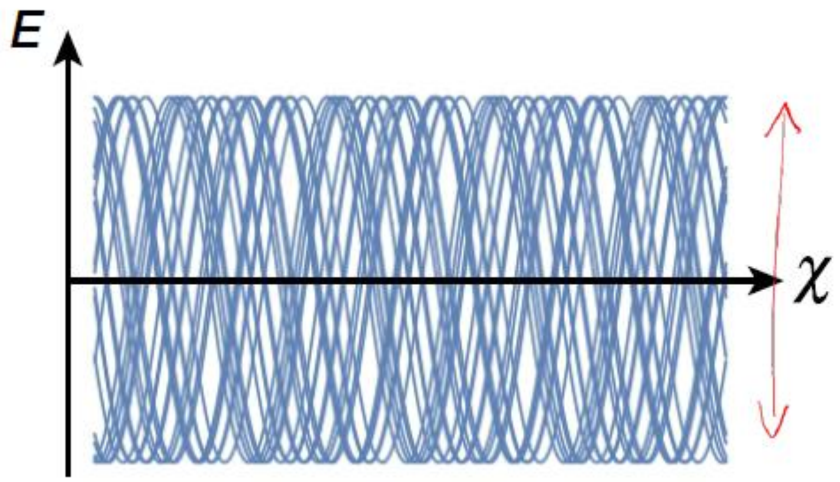
$$\mathcal{E}^{(1)}_{\mathrm{p.w.}} = \sqrt{\frac{\hbar\omega_\ell}{2\epsilon_0 V}}$$

$(\mathrm{n.u}) = (\text{"natural units"})$

Single Radiation Mode - Fock States

Field fluctuations : $Variance$

$$\Delta E(\chi) = (\langle E^2(\chi)\rangle - \langle E(\chi)\rangle^2)^{1/2} = \sqrt{2n+1}\,\mathcal{E}_{\text{p.w.}}^{(1)}$$

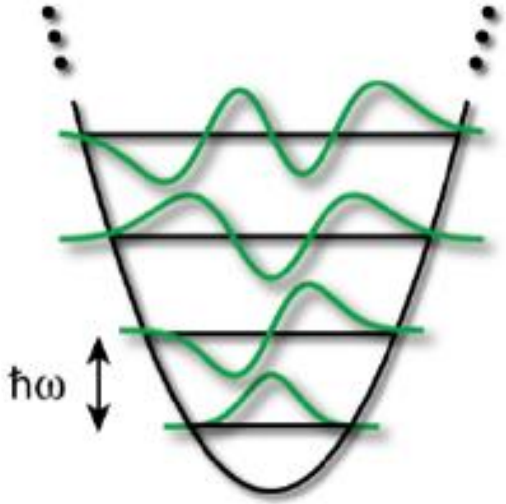


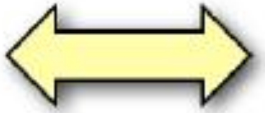
$$\Delta E(\chi) = \sqrt{2n+1}\,\mathcal{E}_{\text{p.w.}}^{(1)}$$

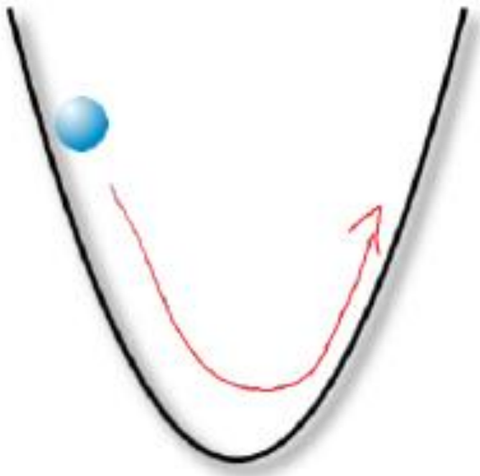
$$\mathcal{E}_{\text{p.w.}}^{(1)} = \sqrt{\frac{\hbar\omega_\ell}{2\epsilon_0 V}}$$

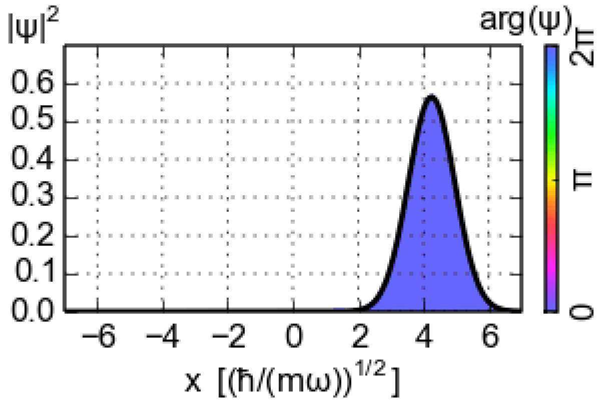
Field states of single radiation field mode: Coherent States

How to reproduce classical motion in a HO









Classical Field States - Coherent States

$$|\alpha\rangle = e^{-\frac{1}{2}|\alpha|^2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle$$

What we want: States of light, whose expectation values corresponds to classical E.M. waves!

Solution: Coherent States: $|\alpha\rangle = |\alpha(0)\rangle = e^{-\frac{1}{2}|\alpha|^2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle$ Heisenberg picture

$$|\alpha(t)\rangle = e^{-i:\hat{H}:t/\hbar}|\alpha(0)\rangle = e^{-\frac{1}{2}|\alpha|^2} \sum_{n=0}^{\infty} \frac{\alpha^n \left(e^{-i\omega t}\right)^n}{\sqrt{n!}} |n\rangle$$

Schrodinger picture

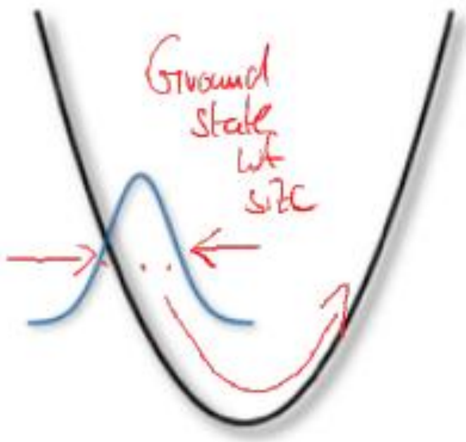
$$= |\alpha(0)e^{-i\omega t}\rangle$$

$$i\hbar \frac{\partial}{\partial t}|\alpha(t)\rangle =: \hat{H}: |\alpha(t)\rangle$$

α **is complex valued:** $\alpha = |\alpha|e^{i\theta}$

How to reproduce classical motion in a HO

$$|\alpha\rangle = e^{-\frac{1}{2}|\alpha|^2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle$$



We adopt a "normal ordered" Hamiltonian for the mode ℓ :

$$:\hat{H}: \; = \hbar\omega \hat{a}^\dagger \hat{a}$$

Superposition of Fock states reproduces oscillating wave-packet motion !

$$|\alpha(t)\rangle = e^{-i:\hat{H}:t/\hbar} |\alpha(0)\rangle \propto |0\rangle + e^{-i\omega t}\alpha|1\rangle + e^{-i2\omega t}\frac{\alpha^2}{\sqrt{2}}|2\rangle + \cdots$$

$$|\alpha(t)\rangle = e^{-\frac{1}{2}|\alpha|^2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} e^{-\frac{iE_n t}{\hbar}} |n\rangle = e^{-\frac{1}{2}|\alpha|^2} \sum_{n=0}^{\infty} \frac{\alpha^n e^{-in\omega t}}{\sqrt{n!}} |n\rangle = |\alpha e^{-i\omega t}\rangle$$

Properties of Coherent States

$$|\alpha\rangle = e^{-\frac{1}{2}|\alpha|^2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle$$

i) $|\alpha\rangle$ are eigenstates of the destruction operator $\hat{a}$: $\hat{a}|\alpha\rangle = \alpha|\alpha\rangle$

Demonstration :

$$\hat{a}|\alpha\rangle = \hat{a} e^{-\frac{1}{2}|\alpha|^2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle = e^{-\frac{1}{2}|\alpha|^2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} \hat{a}|n\rangle$$

$$= e^{-\frac{1}{2}|\alpha|^2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} \sqrt{n}|n-1\rangle = e^{-\frac{1}{2}|\alpha|^2} \alpha \sum_{n=1}^{\infty} \frac{\alpha^{n-1}}{\sqrt{(n-1)!}} |n-1\rangle$$

$$m \equiv n - 1$$

$$= \alpha e^{-\frac{1}{2}|\alpha|^2} \sum_{m=0}^{\infty} \frac{\alpha^m}{\sqrt{m!}} |m\rangle = \alpha|\alpha\rangle$$

Properties of Coherent States

$$|\alpha\rangle = e^{-\frac{1}{2}|\alpha|^2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle$$

ii) $|\alpha\rangle$ are normalized

$$\langle\alpha|\alpha\rangle = e^{-|\alpha|^2} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(\alpha^*)^m}{\sqrt{m!}} \frac{\alpha^n}{\sqrt{n!}} \langle m|n\rangle$$

$$\langle m|n\rangle = \delta_{n,m}$$

$$= e^{-|\alpha|^2} \sum_{n=0}^{\infty} \frac{|\alpha|^{2n}}{n!}$$

$$= e^{-|\alpha|^2} e^{|\alpha|^2} = 1$$

$$\boxed{\langle\alpha|\alpha\rangle = 1}$$

Properties of Coherent States

$$|\alpha\rangle = e^{-\frac{1}{2}|\alpha|^2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle$$

iii) $|\alpha\rangle$ are "quasi orthogonal"

$$\langle\alpha|\beta\rangle = e^{-\frac{1}{2}|\alpha|^2} e^{-\frac{1}{2}|\beta|^2} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(\alpha^*)^m}{\sqrt{m!}} \frac{\beta^n}{\sqrt{n!}} \langle m|n\rangle$$

$$\langle m|n\rangle = \delta_{n,m}$$

$$= e^{-\frac{1}{2}|\alpha|^2} e^{-\frac{1}{2}|\beta|^2} \sum_{n=0}^{\infty} \frac{(\alpha^*\beta)^n}{n!} = e^{-\frac{1}{2}|\alpha|^2 - \frac{1}{2}|\beta|^2 + \alpha^*\beta}$$

$$\langle\alpha|\beta\rangle = e^{-\frac{1}{2}|\alpha|^2 - \frac{1}{2}|\beta|^2 + \alpha^*\beta}$$

$$= \exp\left[\frac{1}{2}(\alpha^*\beta - \beta^*\alpha)\right] e^{-\frac{1}{2}[|\alpha|^2 + |\beta|^2 - \alpha^*\beta - \beta^*\alpha]} = \exp\left[\frac{1}{2}(\alpha^*\beta - \beta^*\alpha)\right] e^{-\frac{1}{2}|\alpha-\beta|^2}$$

$$|\langle\alpha|\beta\rangle|^2 = e^{-|\alpha-\beta|^2} \neq 0$$

$$|\alpha - \beta| \gg 1 \Longrightarrow |\langle\alpha|\beta\rangle| \longrightarrow 0$$

Properties of Coherent States

$$|\alpha\rangle = e^{-\frac{1}{2}|\alpha|^2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle$$

iv) Coherent states form a complete set of quantum states

$$\hat{\mathbb{I}} = \int \frac{d^2\alpha}{\pi} |\alpha\rangle\langle\alpha|$$

Demonstration :

$$\int d^2\alpha |\alpha\rangle\langle\alpha| = \int e^{-|\alpha|^2} \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{\alpha^n \alpha^{*,m}}{\sqrt{n!\,m!}} |n\rangle\langle m|\, d^2\alpha$$

$$\begin{cases} \alpha = re^{i\theta} \\ d^2\alpha = rdrd\theta \end{cases}$$

$$= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{|n\rangle\langle m|}{\sqrt{n!\,m!}} \int_0^{\infty} dr\, e^{-r^2} r^{n+m+1} \int_0^{2\pi} d\phi\, e^{i(n-m)\theta}$$

$$\left\{ \int_0^{2\pi} d\theta\, e^{i(n-m)\theta} = 2\pi\delta_{n,m} \right.$$

$$= 2\pi \sum_{n=0}^{\infty} \frac{|n\rangle\langle n|}{n!} \int_0^{\infty} rdr\, e^{-r^2} r^{2n} = \pi \sum_{n=0}^{\infty} \frac{|n\rangle\langle n|}{n!} \int_0^{\infty} dy\, e^{-y} y^n$$

$$\begin{cases} y = r^2 \\ dy = 2rdr \end{cases}$$

$$= \pi \sum_{n=0}^{\infty} |n\rangle\langle n| = \pi\hat{\mathbb{I}}$$

$$\int_0^{\infty} dy\, e^{-y} y^n = n!$$

Properties of Coherent States

$$|\alpha\rangle = e^{-\frac{1}{2}|\alpha|^2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle$$

v) Average photon number : $I \propto \bar{n} = \langle n \rangle = \langle \alpha | \hat{a}^\dagger \hat{a} | \alpha \rangle = |\alpha|^2 \langle \alpha | \alpha \rangle = |\alpha|^2$

Properties of Coherent States

$$|\alpha\rangle = e^{-\frac{1}{2}|\alpha|^2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle$$

vi) Photon number variance

$$(\Delta n)^2 = \langle\alpha| \left(\hat{N} - \langle\hat{N}\rangle\right)^2 |\alpha\rangle = \langle\alpha|\, \hat{N}^2 |\alpha\rangle - \langle\hat{N}\rangle^2$$

$$= \langle\alpha|\, \hat{N}^2 |\alpha\rangle - |\alpha|^4$$

$$= \langle\alpha|\, \hat{a}^\dagger \hat{a} \hat{a}^\dagger \hat{a} |\alpha\rangle - |\alpha|^4$$

$$\hat{a}\hat{a}^\dagger = 1 + \hat{a}^\dagger \hat{a}$$

$$= \langle\alpha|\, \hat{a}^\dagger \hat{a}^\dagger \hat{a} \hat{a} |\alpha\rangle + \langle\alpha| \hat{a}^\dagger \hat{a} |\alpha\rangle - |\alpha|^4$$

$$= |\alpha|^4 + |\alpha|^2 - |\alpha|^4$$

$$= |\alpha|^2 = \bar{n}$$

$$\Delta n = |\alpha| = \sqrt{\bar{n}}$$

Properties of Coherent States

$$|\alpha\rangle = e^{-\frac{1}{2}|\alpha|^2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle$$

vii) Photon number distribution

$$P(n) = |\langle n|\alpha\rangle|^2$$

$$= \left| \langle n| e^{-\frac{1}{2}|\alpha|^2} \sum_{m=0}^{\infty} \frac{\alpha^m}{\sqrt{m!}} |m\rangle \right|^2$$

$$= \left| e^{-\frac{1}{2}|\alpha|^2} \sum_{m=0}^{\infty} \frac{\alpha^m}{\sqrt{m!}} \delta_{n,m} \right|^2$$

$$\langle n|m\rangle = \delta_{n,m}$$

$$= e^{-|\alpha|^2} \frac{\alpha^{2n}}{n!}$$

$$= e^{-\bar{n}} \frac{\bar{n}^n}{n!}$$

$$\bar{n} = |\alpha|^2$$

Poisson distribution of the number of photons

Properties of Coherent States

$$|\alpha\rangle = e^{-\frac{1}{2}|\alpha|^2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle$$

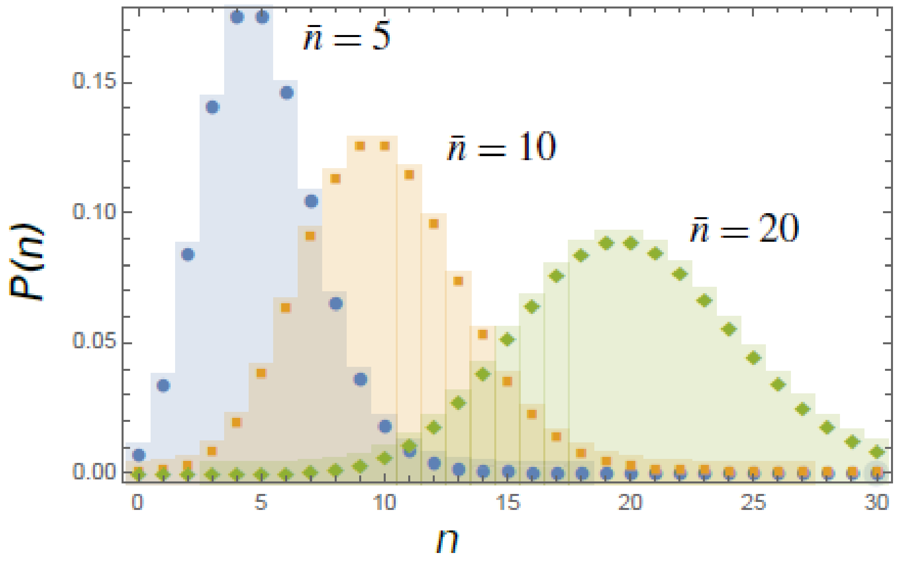
vi) Photon number distribution

$$P(n) = |\langle n|\alpha\rangle|^2 = e^{-\bar{n}} \frac{\bar{n}^n}{n!}$$

Poisson distribution !

$$\Delta n = |\alpha| = \sqrt{\bar{n}}$$

$$\frac{\Delta n}{\bar{n}} = \frac{1}{\sqrt{\bar{n}}}$$



Properties of Coherent States

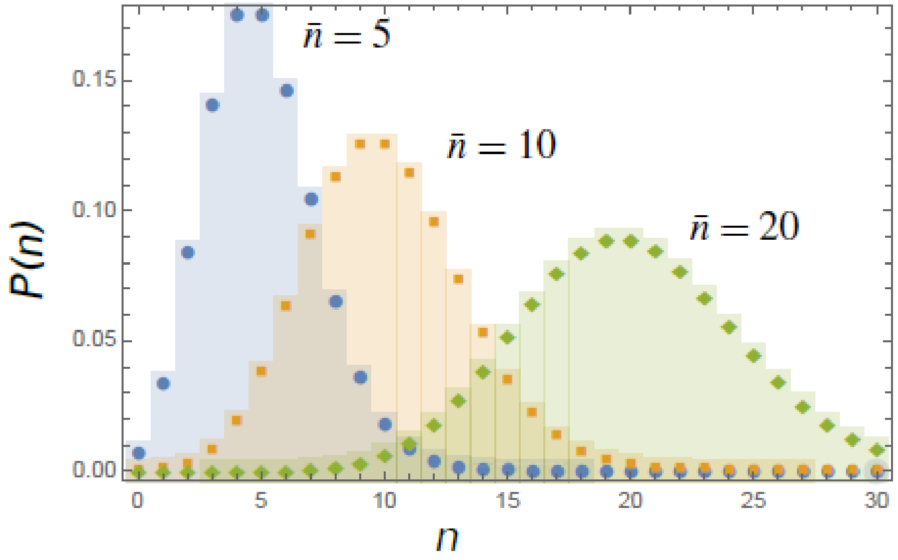
$$|\alpha\rangle = e^{-\frac{1}{2}|\alpha|^2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle$$

vi) Photon number distribution

$$P(n) = |\langle n|\alpha\rangle|^2 = e^{-\bar{n}} \frac{\bar{n}^n}{n!}$$

Poisson distribution

$$\frac{\Delta n}{\bar{n}} = \frac{1}{\sqrt{\bar{n}}}$$



For large $\bar{n}$

$$P(n) \cong \frac{1}{\sqrt{2\pi\bar{n}}} e^{-\frac{1}{2}\frac{(n-\bar{n})^2}{\bar{n}}}$$

Gaussian distribution

Properties of Coherent States

$$|\alpha\rangle = e^{-\frac{1}{2}|\alpha|^2}\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}|n\rangle$$

vii) Average Electric field of a coherent state

$$\langle E(\chi)\rangle = \langle\alpha|\hat{E}(\chi)|\alpha\rangle = \langle\alpha|\hat{a}e^{-i\chi} + \hat{a}^\dagger e^{i\chi}|\alpha\rangle$$

$$\alpha = |\alpha|e^{i\theta}$$

$$\chi_\ell \equiv \omega_\ell t - \boldsymbol{k}_\ell\cdot\boldsymbol{r} - \frac{\pi}{2}$$

$$= \langle\alpha|\alpha e^{-i\chi} + e^{i\chi}\alpha^*|\alpha\rangle$$

$$= |\alpha|\langle\alpha|e^{i\theta}e^{-i\chi} + e^{i\chi}e^{-i\theta}|\alpha\rangle$$

$$= |\alpha|\left(e^{-i(\chi-\theta)} + e^{i(\chi-\theta)}\right)$$

$$= 2|\alpha|\cos(\chi-\theta)(\text{n.u.}) = 2|\alpha|\cos(\chi-\theta)\mathcal{E}^{(1)}_{\text{p.w.}}$$

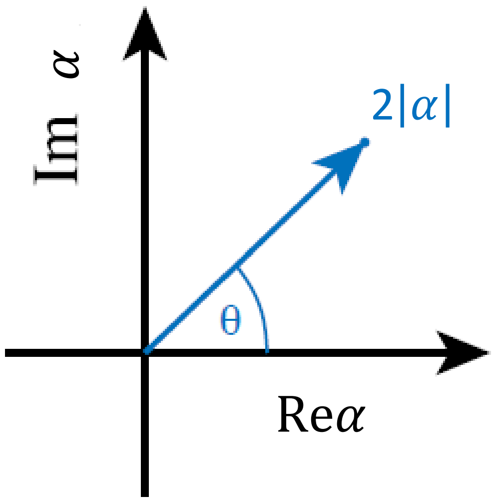
$$\mathcal{E}^{(1)}_{\text{p.w.}} = \sqrt{\frac{\hbar\omega_\ell}{2\epsilon_0 V}}$$

Properties of Coherent States

$$|\alpha\rangle = e^{-\frac{1}{2}|\alpha|^2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle$$

vii) Average electric field of a coherent state

$$\langle E(\chi)\rangle = \langle\alpha|\hat{E}(\chi)|\alpha\rangle = 2|\alpha|\cos(\chi - \theta) \ \text{(n.u.)} \qquad \alpha = |\alpha|e^{i\theta} \qquad \chi \equiv \omega t - i\boldsymbol{k}\cdot\boldsymbol{r} - \frac{\pi}{2}$$



Field representation in a **Phasor Diagram**

Properties of Coherent States

$$|\alpha\rangle = e^{-\frac{1}{2}|\alpha|^2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle$$

viii) E-field fluctuations (variance)

$$\left(\Delta E(\chi)\right)^2 = \langle\alpha|\hat{E}^2(\chi)|\alpha\rangle - \left|\langle\hat{E}(\chi)\rangle\right|^2$$

$$|\langle E(\chi)\rangle|^2 = 4|\alpha|^2\cos^2(\chi-\theta)$$

$$\langle\alpha|\hat{E}^2(\chi)|\alpha\rangle = \langle\alpha|\left(\hat{a}e^{-i\chi} + \hat{a}^\dagger e^{i\chi}\right)^2|\alpha\rangle$$

$$[\hat{a},\hat{a}^\dagger] = 1 \Rightarrow \hat{a}\hat{a}^\dagger = 1 + \hat{a}^\dagger\hat{a}$$

$$= \langle\alpha|\left(\hat{a}^2 e^{-2i\chi} + \left(\hat{a}^\dagger\right)^2 e^{2i\chi} + \hat{a}\hat{a}^\dagger + \hat{a}^\dagger\hat{a}\right)|\alpha\rangle$$

$$= \langle\alpha|\hat{a}^2 e^{-2i\chi} + \left(\hat{a}^\dagger\right)^2 e^{2i\chi} + 1 + 2\hat{a}^\dagger\hat{a}|\alpha\rangle$$

$$= \alpha^2 e^{-2i\chi} + (\alpha^*)^2 e^{2i\chi} + 2|\alpha|^2 + 1$$

$$= \left(\alpha e^{-i\chi} + \alpha^* e^{i\chi}\right)^2 + 1 = |\alpha|\left(e^{-i(\chi-\theta)} + e^{i(\chi-\theta)}\right)^2 + 1$$

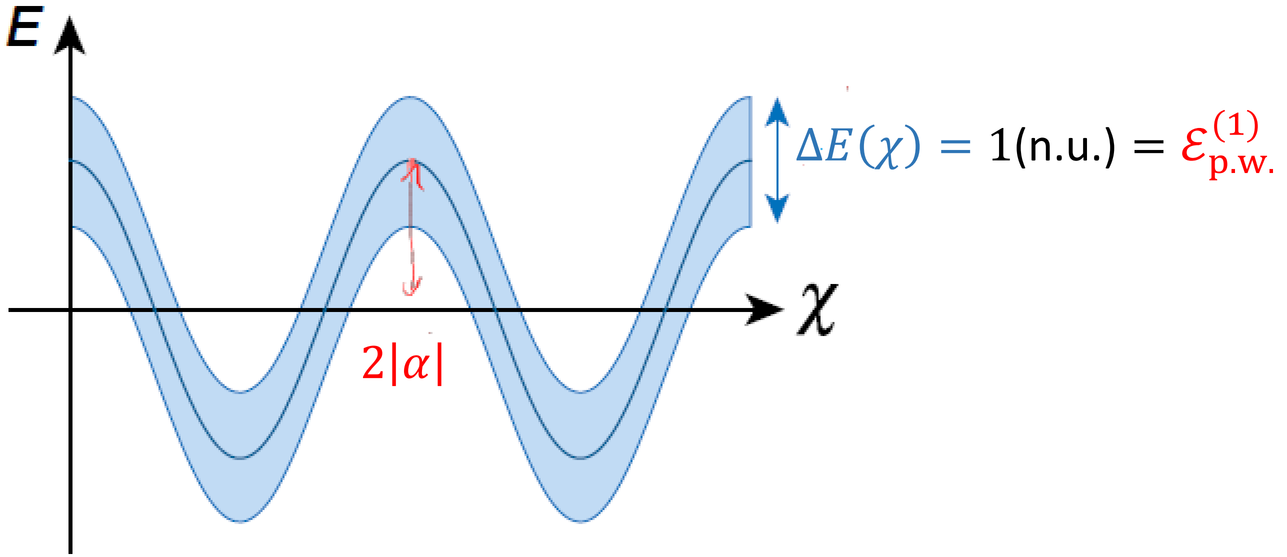
$$= 4|\alpha|^2\cos^2(\chi-\theta) + 1$$

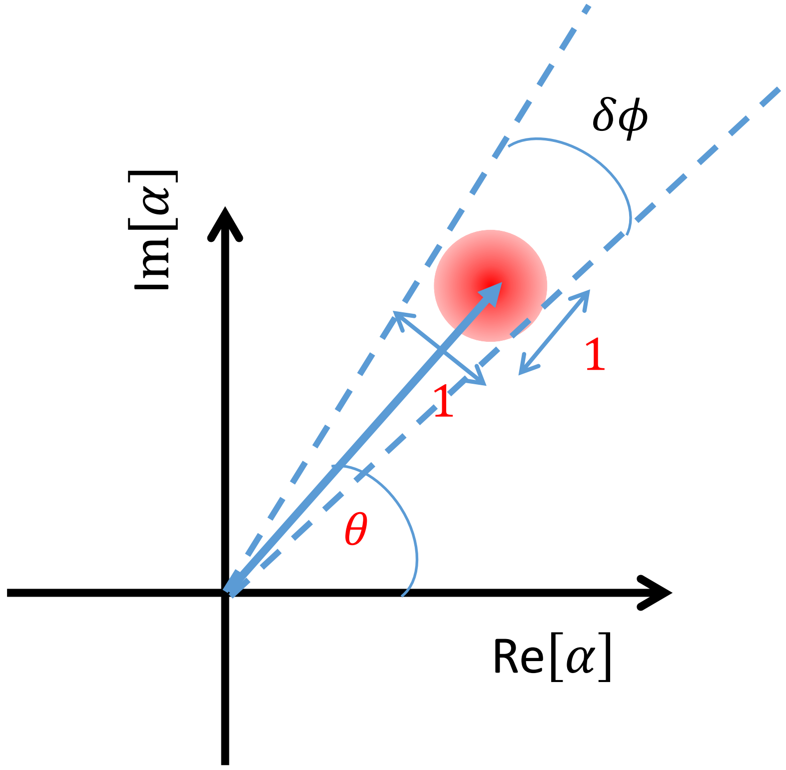
$$\Delta E(\chi) = 1(\text{n.u.}) = \mathcal{E}_{\text{p.w.}}^{(1)}$$

Properties of Coherent States

$$|\alpha\rangle = e^{-\frac{1}{2}|\alpha|^2}\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}|n\rangle$$

viii) E-field fluctuations (variance) $\Delta E(\chi) = 1(\text{n.u.}) = \mathcal{E}^{(1)}_{\text{p.w.}}$

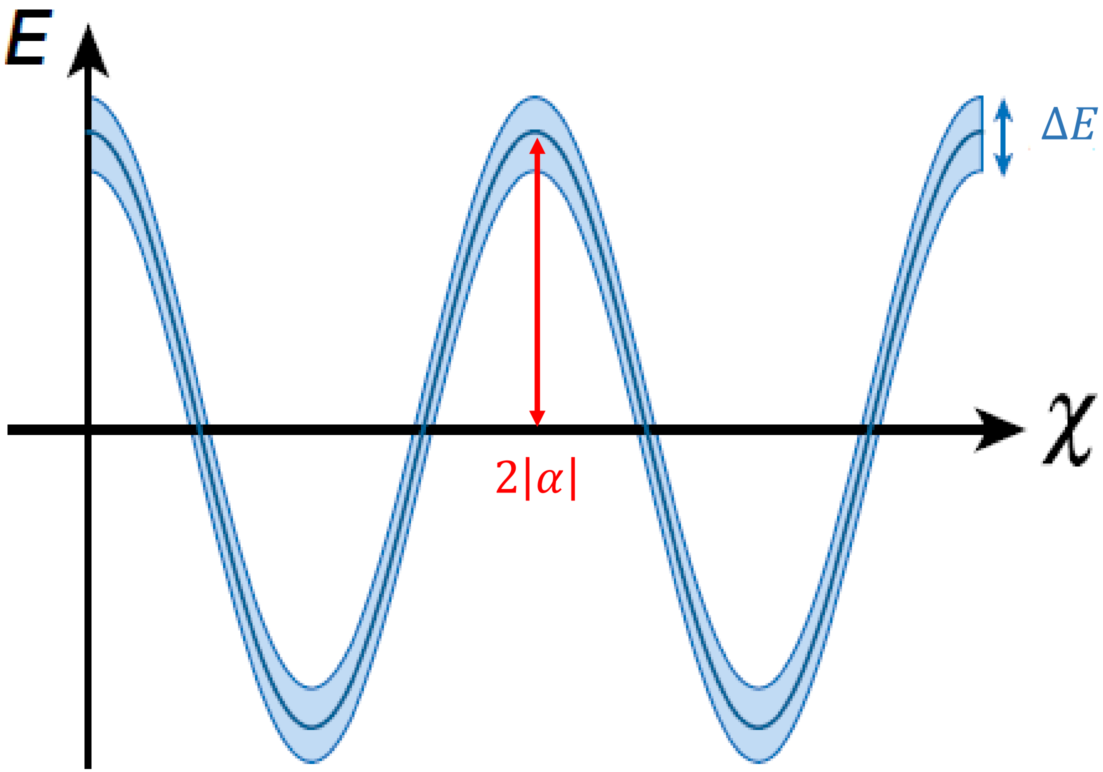


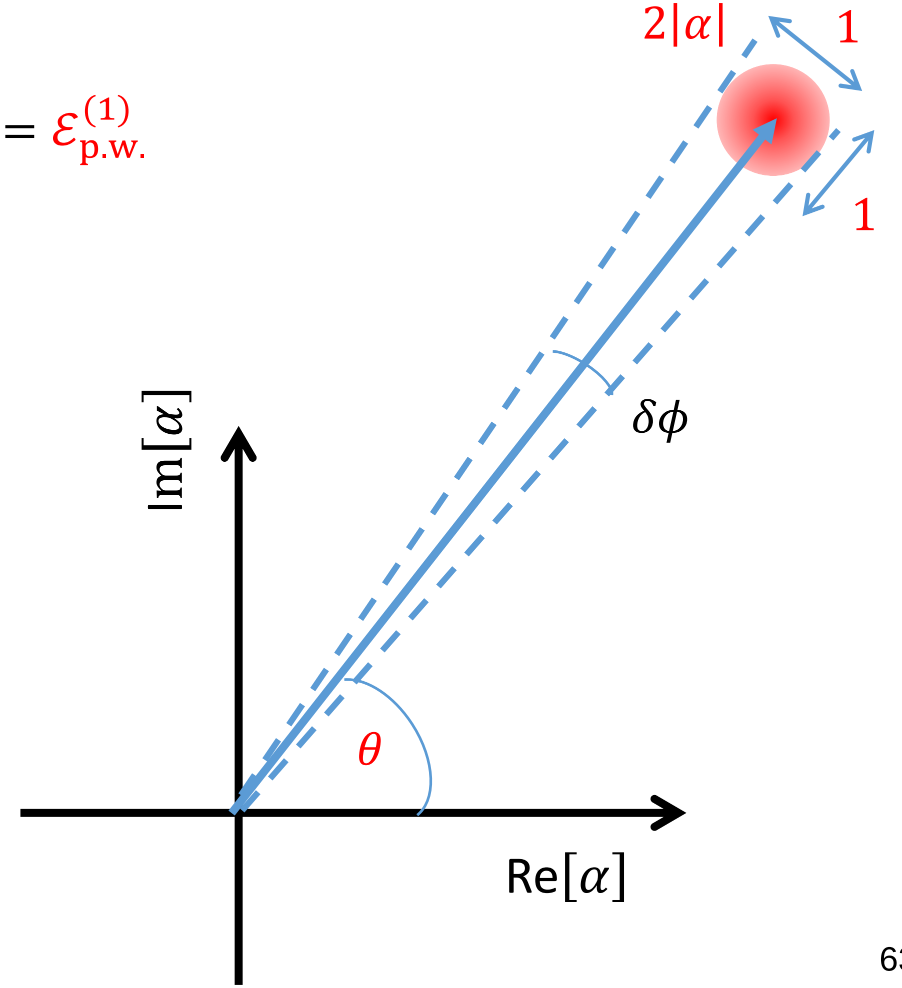


Properties of Coherent States

$$|\alpha\rangle = e^{-\frac{1}{2}|\alpha|^2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle$$

viii) E-field fluctuations (variance) $\Delta E(\chi) = 1(\text{n.u.}) = \mathcal{E}^{(1)}_{\text{p.w.}}$





Properties of Coherent States

$$|\alpha\rangle = e^{-\frac{1}{2}|\alpha|^2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle$$

ix) Energy of coherent states

$$\langle H \rangle = \langle \alpha | : \hat{H} : | \alpha \rangle = \hbar\omega \langle \alpha | \hat{N} | \alpha \rangle$$

$$\hat{N} = \hat{a}^\dagger \hat{a}$$

$$= \hbar\omega \langle \alpha | |\alpha|^2 | \alpha \rangle = \hbar\omega |\alpha|^2$$

$$\boxed{\langle H \rangle = \hbar\omega\bar{n}}$$

$$\bar{n} = |\alpha|^2 \propto I$$

Properties of Coherent States

$$|\alpha\rangle = e^{-\frac{1}{2}|\alpha|^2}\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}|n\rangle$$

ix) Fluctuation of of coherent state energy

$$(\Delta H)^2 = \hbar^2\omega^2\langle\alpha|\left(\hat{N} - \langle\hat{N}\rangle\right)^2|\alpha\rangle = \langle\alpha|\,\hat{N}^2|\alpha\rangle - \langle\alpha|\,\hat{N}|\alpha\rangle$$

$$= \hbar^2\omega^2\bar{n} = \hbar^2\omega^2|\alpha|^2$$

$$\hat{N} = \hat{a}^\dagger\hat{a}$$

$$\langle H\rangle = \hbar\omega\left(\bar{n} + \frac{1}{2}\right)$$

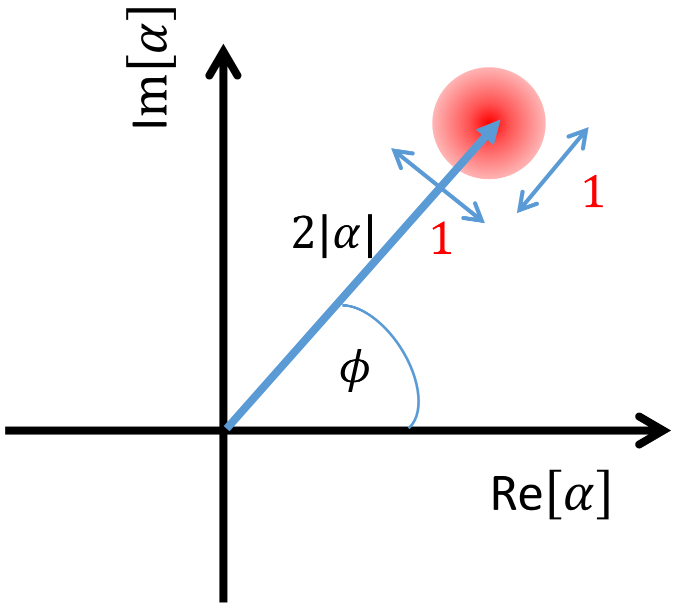
$$\Delta H = \hbar\omega|\alpha|$$

$$\frac{\Delta H}{\langle H\rangle} = \frac{1}{|\alpha|} = \frac{1}{\sqrt{\bar{n}}}$$

Coherent state - a displaced vacuum

Displacement Operator (shifts any coherent state by α)

$$\hat{D}(\alpha) \equiv e^{\alpha \hat{a}^\dagger - \alpha^* \hat{a}} \qquad \hat{D}(\alpha)|\beta\rangle \rightarrow |\alpha+\beta\rangle$$



Coherent State is a "displaced" vacuum state: $|\alpha\rangle = \hat{D}(\alpha)|0\rangle$

$$\hat{D}(\alpha) \equiv e^{\alpha \hat{a}^\dagger - \alpha^* \hat{a}} = e^{\hat{A}+\hat{B}} \qquad \hat{A} \equiv \alpha \hat{a}^\dagger \qquad \hat{B} \equiv -\alpha^* \hat{a} \qquad [\hat{A},\hat{B}] = -|\alpha|^2[\hat{a}^\dagger, \hat{a}] = |\alpha|^2$$

We need the disentangling theorem : $\hat{D}(\alpha) \equiv e^{\hat{A}+\hat{B}} = e^{\hat{A}} e^{\hat{B}} e^{-\frac{1}{2}[\hat{A},\hat{B}]} = e^{\hat{B}} e^{\hat{A}} e^{\frac{1}{2}[\hat{A},\hat{B}]}$

$$\hat{D}(\alpha) = e^{\alpha \hat{a}^\dagger - \alpha^* \hat{a}} = e^{-\frac{1}{2}|\alpha|^2} e^{\alpha \hat{a}^\dagger} e^{-\alpha^* \hat{a}} \qquad e^{-\alpha^* \hat{a}}|0\rangle = |0\rangle$$

And $(\hat{a}^\dagger)^n|0\rangle = \sqrt{n!}\,|n\rangle$

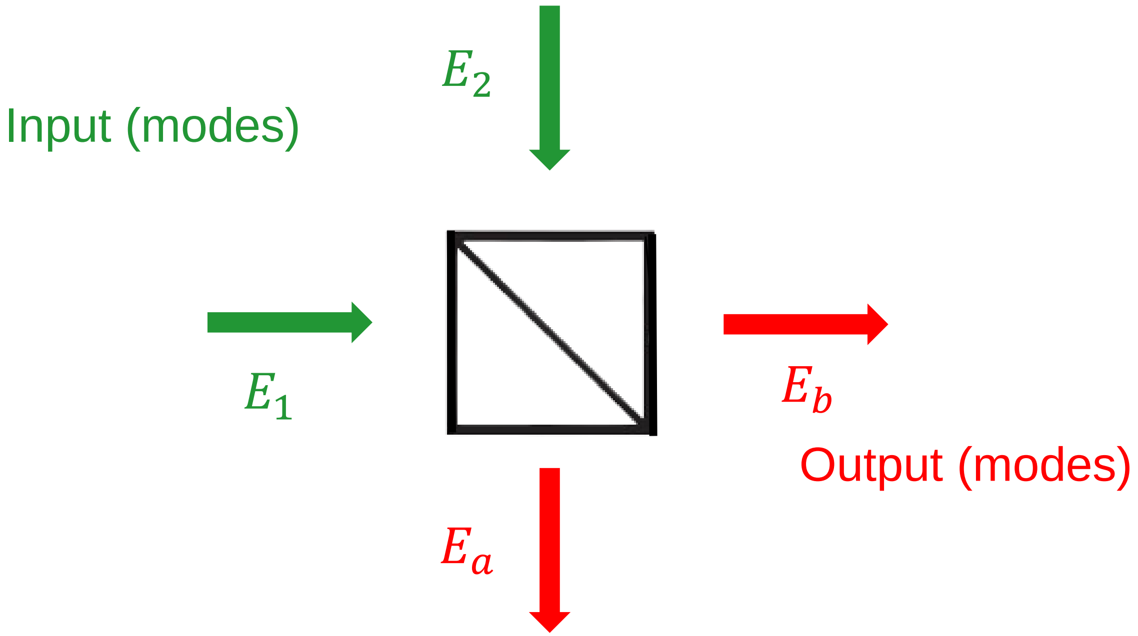
$$\longrightarrow \quad \hat{D}(\alpha)\,|0\rangle = e^{-\frac{1}{2}|\alpha|^2} e^{\alpha \hat{a}^\dagger}|0\rangle = e^{-\frac{1}{2}|\alpha|^2} \sum_{n=0}^{\infty} \frac{\alpha^n (\hat{a}^\dagger)^n}{n!}|0\rangle = e^{-\frac{1}{2}|\alpha|^2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}}|n\rangle = |\alpha\rangle$$

The classical beam-splitter

(energy conservation constraints)

The `classical' beam-splitter

We assume that all beams have the same polarization and frequency



(no losses, idealized BS)

$$E_a = RE_1 + TE_2$$

$$E_b = T'E_1 + R'E_2$$

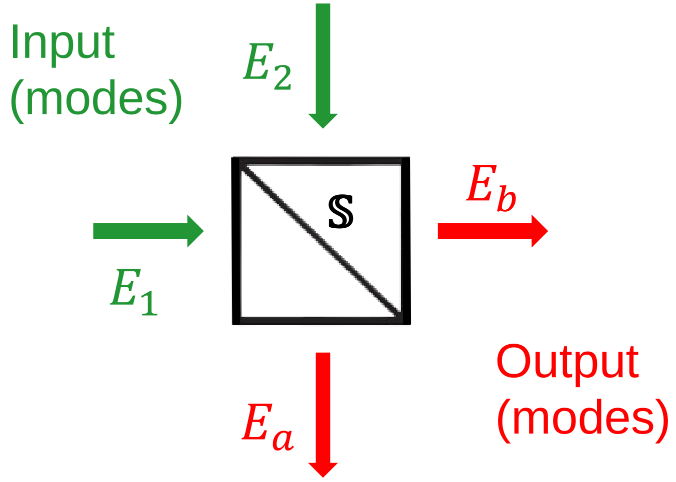
$$\begin{pmatrix} E_a \\ E_b \end{pmatrix} = \begin{pmatrix} R & T \\ T' & R' \end{pmatrix} \begin{pmatrix} E_1 \\ E_2 \end{pmatrix}$$

$\mathbb{S} \equiv \begin{pmatrix} R & T \\ T' & R' \end{pmatrix}$ is a general S-matrix

Symmetric beam-splitter S-matrix : $T' \rightarrow T$, $R' \rightarrow R$

$$\overset{\mathbb{S}}{}$$

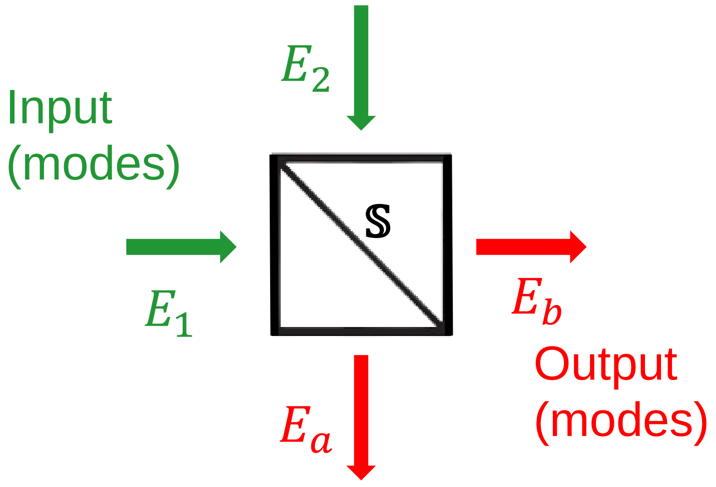
$$\begin{pmatrix} E_a \\ E_b \end{pmatrix} = \underbrace{\begin{pmatrix} R & T \\ T & R \end{pmatrix}}_{\mathbb{S}} \begin{pmatrix} E_1 \\ E_2 \end{pmatrix}$$



Energy conservation in a symmetric beam-splitter

$$I \propto |\boldsymbol{\Pi}|^2 \propto |E|^2$$

$$\boldsymbol{\Pi} = \frac{1}{2}\mathrm{Re}\{\boldsymbol{E}^* \times \boldsymbol{H}\}$$



$$I_1 = I_a + I_b$$

$$|E_1|^2 = |E_a|^2 + |E_b|^2 = (|R|^2 + |T|^2)|E_1|^2 \Longrightarrow |R|^2 + |T|^2 = 1$$

$$\begin{pmatrix} E_a \\ E_b \end{pmatrix} = \begin{pmatrix} R & T \\ T & R \end{pmatrix} \begin{pmatrix} E_1 \\ E_2 \end{pmatrix}$$

$$\equiv \mathbb{S} \begin{pmatrix} E_1 \\ E_2 \end{pmatrix}$$

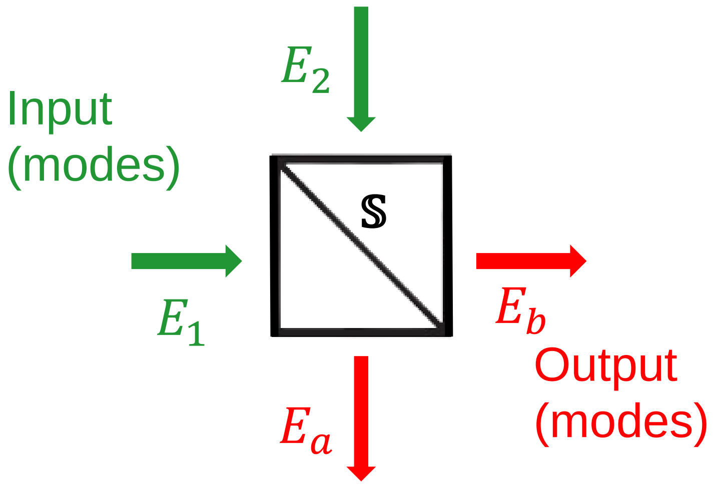
$$I_2 = I_a' + I_b'$$

$$|E_2|^2 = |E'_a|^2 + |E'_b|^2 = (|R|^2 + |T|^2)|E_2|^2 \Longrightarrow |R|^2 + |T|^2 = 1$$

Energy conservation in a symmetric beam-splitter

$$I_1 + I_2 = I_a + I_b$$

$$|R|^2 + |T|^2 = 1$$



$$|E_1|^2 + |E_2|^2 = |E_a|^2 + |E_b|^2$$

$$= (R^* E_1^* + T^* E_2^*)(R E_1 + tT) + (T^* E_1^* + R^* E_2^*)(T E_1 + R E_2)$$

$$= |E_1|^2 + |E_2|^2 + (R^* T + T^* R) E_1^* E_2 + (T R^* + T^* R) E_1 E_2^*$$

$$\begin{pmatrix} E_a \\ E_b \end{pmatrix} = \begin{pmatrix} R & T \\ T & R \end{pmatrix} \begin{pmatrix} E_1 \\ E_2 \end{pmatrix} \equiv \mathbb{S} \begin{pmatrix} E_1 \\ E_2 \end{pmatrix}$$

$$\Rightarrow \quad R^* T + T^* R = 0$$

$$E_a = R E_1 + T E_2$$

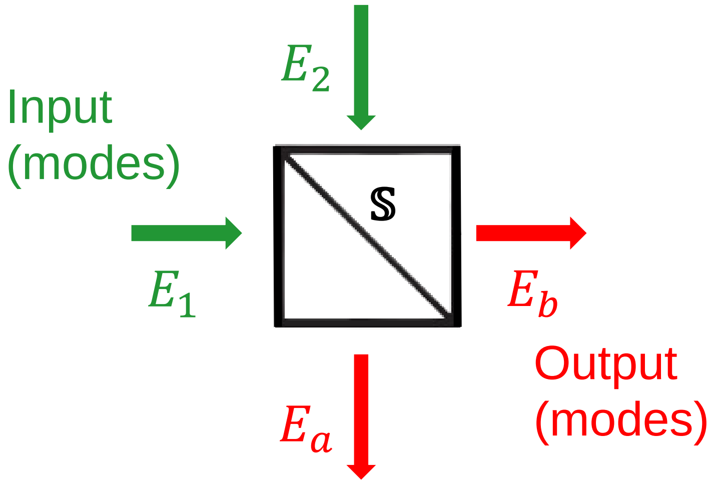
$$E_b = T T + R E_2$$

Full energy conservation ⟶ unitary S-matrix

$$\mathbb{S}^\dagger = \mathbb{S}^{-1} \quad \Longrightarrow \quad \begin{pmatrix} R^* & T^* \\ T & R^* \end{pmatrix}\begin{pmatrix} R & T \\ T & R \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$|R|^2 + |T|^2 = 1$$

$$T^*R + R^*T = 0$$



$$\begin{pmatrix} E_a \\ E_b \end{pmatrix} = \begin{pmatrix} R & T \\ T & R \end{pmatrix}\begin{pmatrix} E_1 \\ E_2 \end{pmatrix} \equiv \mathbb{S}\begin{pmatrix} E_1 \\ E_2 \end{pmatrix}$$

Let us write: $R = |R|e^{i\phi_R}$ and : $T = |T|e^{i\phi_T}$

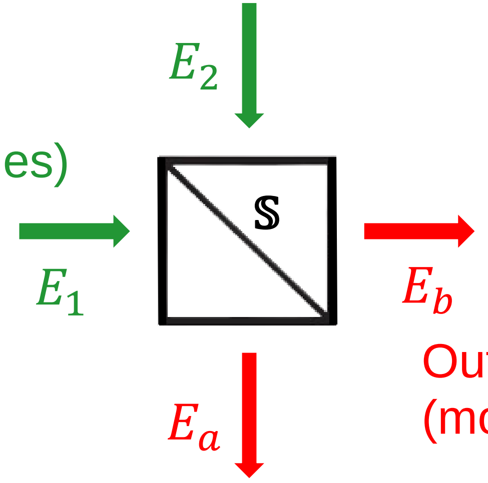
$$|T||R|e^{i(\phi_R-\phi_T)} + |T||R|e^{-i(\phi_R-\phi_T)} = 0 \Longrightarrow \cos(\phi_R - \phi_T) = 0 \Longrightarrow \phi_R - \phi_T = \frac{\pi}{2}$$

Set $\phi_T = 0$ ⟹ $\phi_R = \frac{\pi}{2}$, $e^{i\frac{\pi}{2}} = i$ ⟹ $T \longrightarrow |T| \quad R \longrightarrow i|R|$

50/50 symmetric beam-splitter

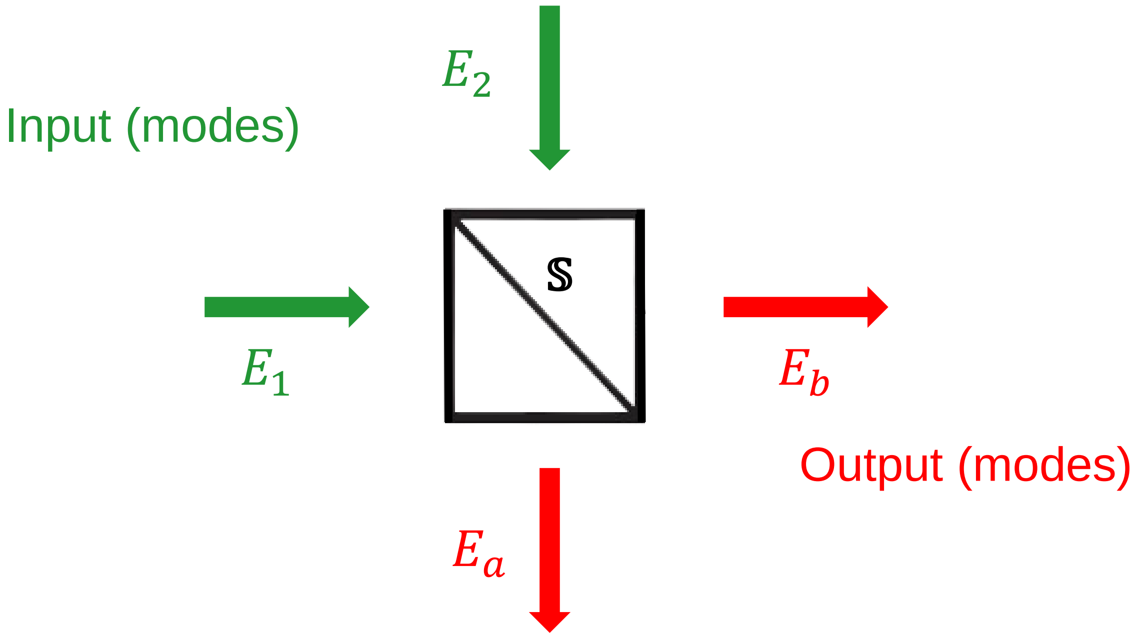
Symmetrized 50/50 beam-splitter: $|R| = |T| = \frac{1}{\sqrt{2}}$ $\quad R = iT$

$$\overbrace{\begin{pmatrix} E_a \\ E_b \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} i & 1 \\ 1 & i \end{pmatrix}\begin{pmatrix} E_1 \\ E_2 \end{pmatrix}}^{\mathbb{S}}$$



$$\underbrace{\frac{1}{\sqrt{2}}\begin{pmatrix} -i & 1 \\ 1 & -i \end{pmatrix}}_{\mathbb{S}^\dagger} \cdot \underbrace{\frac{1}{\sqrt{2}}\begin{pmatrix} i & 1 \\ 1 & i \end{pmatrix}}_{\mathbb{S}} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Non-symmetric beam-splitter



(no losses, idealized BS)

$$E_a = RE_1 + TE_2$$

$$E_b = T'E_1 + R'E_2$$

$$\begin{pmatrix} E_a \\ E_b \end{pmatrix} = \begin{pmatrix} R & T \\ T' & R' \end{pmatrix} \begin{pmatrix} E_1 \\ E_2 \end{pmatrix}$$

$\mathbb{S} \equiv \begin{pmatrix} R & T \\ T' & R' \end{pmatrix}$ general form of the S-matrix

Non-symmetric 50/50 beam-splitter

50/50 beam-splitter: $R = -R' = T = T' = \frac{1}{\sqrt{2}}$

$$\mathbb{S} = \mathbb{S}^{\dagger} = \mathbb{S}^{-1}$$

$$\begin{pmatrix} E_a \\ E_b \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} E_1 \\ E_2 \end{pmatrix}$$

$\mathbb{S}$ $\quad\quad$ $\mathbb{S}^{\dagger}$

$$\frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Justification for this choice : Partially silvered mirror?

Non-symmetrized 50/50 beam-splitter: $R = -R' = T = T' = \frac{1}{\sqrt{2}}$

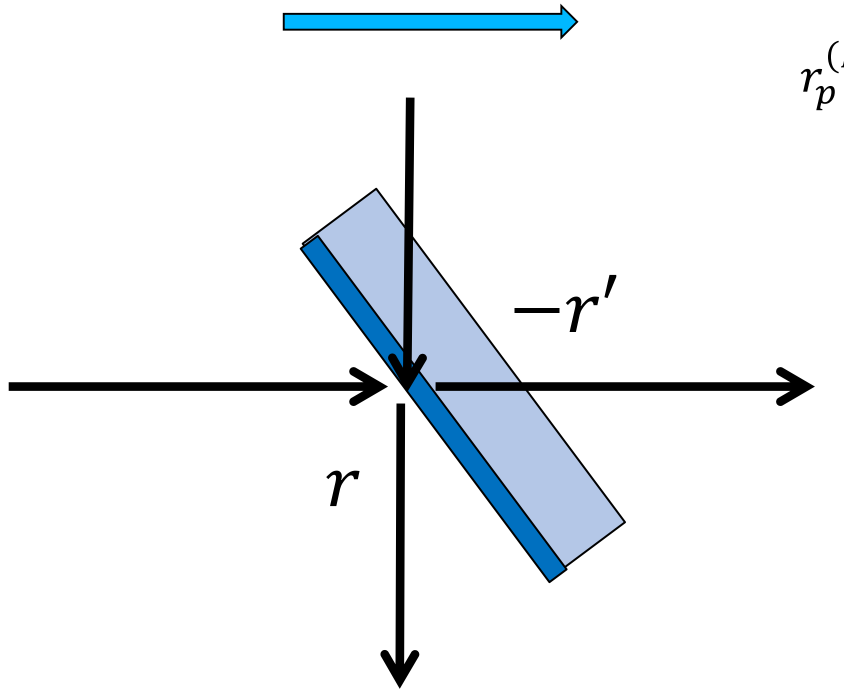
$$r_s^{(F)} = r_\perp^{(F)} = \frac{n_1 \cos\theta_\mathrm{i} - n_2 \cos\theta_\mathrm{t}}{n_1 \cos\theta_\mathrm{i} + n_2 \cos\theta_\mathrm{t}}$$

$$r_p^{(F)} = r_\parallel^{(F)} = \frac{n_2 \cos\theta_\mathrm{i} - n_1 \cos\theta_\mathrm{t}}{n_1 \cos\theta_\mathrm{t} + n_2 \cos\theta_\mathrm{i}}$$

$$\theta_\mathrm{i} = 0$$

$$r_s^{(F)} = r_\perp^{(F)} = \frac{n_1 - n_2}{n_1 + n_2}$$

$$r_p^{(F)} = r_\parallel^{(F)} = \frac{n_2 - n_1}{n_1 + n_2}$$



$\mathbb{S}$-matrix for a "non-mirror symmetric" 50/50 beam-splitter

50/50 beam-splitter: $R = -R' = T = T' = \frac{1}{\sqrt{2}}$

$$\mathbb{S} = \mathbb{S}^\dagger = \mathbb{S}^{-1}$$

$$\begin{pmatrix} E_a \\ E_b \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} E_1 \\ E_2 \end{pmatrix}$$

$\mathbb{S}$ $\qquad$ $\mathbb{S}^\dagger$

$$\frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

General form for a unitary 2×2 S-matrix

Beam-splitter, Λ , ψ, ϕ, R , $T \in \mathbb{R}$ $\qquad R^2 + T^2 = 1$

$$\mathbb{S} = e^{i\Lambda/2}\begin{pmatrix} e^{i\psi/2} & 0 \\ 0 & e^{-i\psi/2} \end{pmatrix}\begin{pmatrix} R & T \\ T & -R \end{pmatrix}\begin{pmatrix} e^{i\phi/2} & 0 \\ 0 & e^{-i\phi/2} \end{pmatrix}$$

$$= e^{i\Lambda/2}\begin{pmatrix} e^{i\psi/2}Re^{i\phi/2} & e^{i\psi/2}Te^{-i\phi/2} \\ e^{-i\psi/2}Te^{i\phi/2} & -e^{-i\psi/2}Re^{-i\phi/2} \end{pmatrix} \equiv e^{i\Lambda/2}\begin{pmatrix} R' & T' \\ T'^* & -R'^* \end{pmatrix}$$

$$\mathbb{S}^\dagger.\mathbb{S} = \begin{pmatrix} R'^* & T' \\ T'^* & -R' \end{pmatrix}\begin{pmatrix} R' & T' \\ T'^* & -R'^* \end{pmatrix} = \begin{pmatrix} |R|^2 + |T|^2 & R'^*T' - T'R'^* \\ T'^*R' - R'T'^* & |R|^2 + |T|^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

General form for a unitary 2×2 S-matrix

Beam-splitter, $\Lambda = 0\,,\, \psi = \phi = \frac{\pi}{2},\, R\,,\, T \in \mathbb{R}$ $\quad R^2 + T^2 = 1$

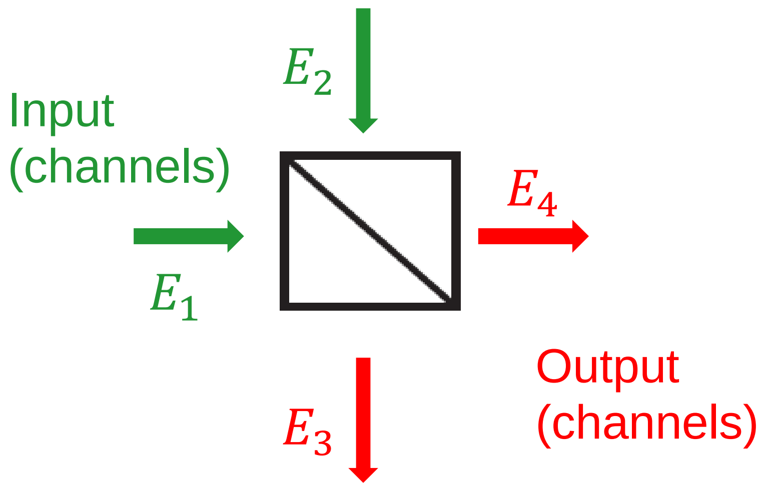
$$\mathbb{S} = e^{i\Lambda/2} \begin{pmatrix} e^{i\pi/4} & 0 \\ 0 & e^{-i\pi/4} \end{pmatrix} \begin{pmatrix} R & T \\ T & -R \end{pmatrix} \begin{pmatrix} e^{i\pi/4} & 0 \\ 0 & e^{-i\pi/4} \end{pmatrix}$$

$$= \begin{pmatrix} iR & T \\ T & iR \end{pmatrix}$$

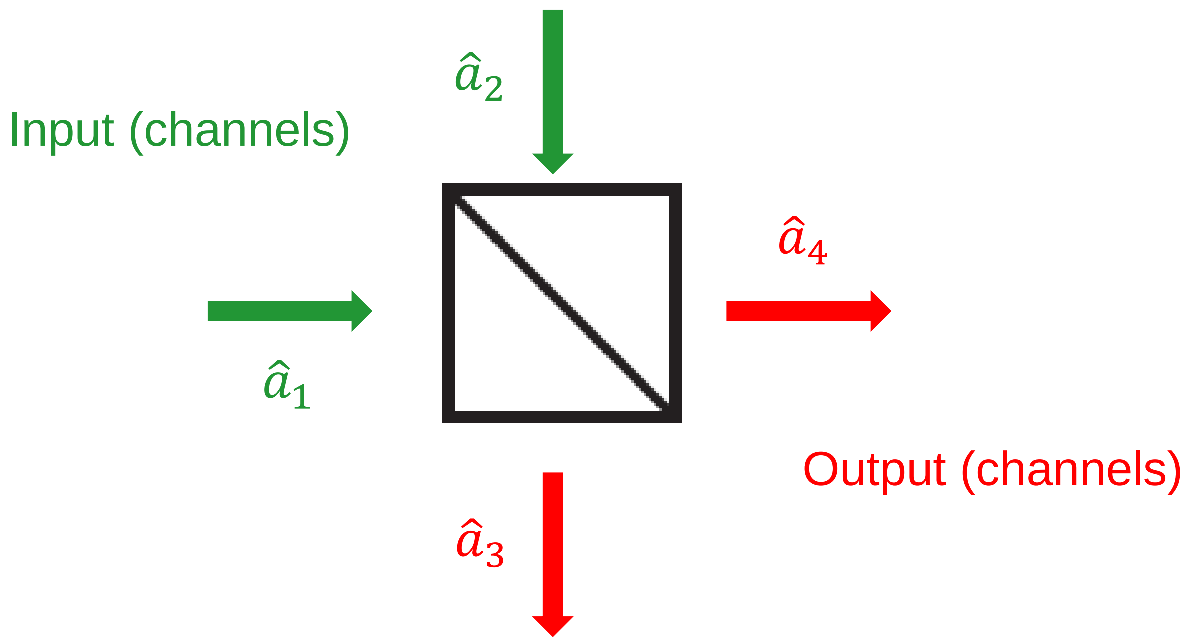
The quantum beam-splitter

Classical vs quantum

Input-output relations:



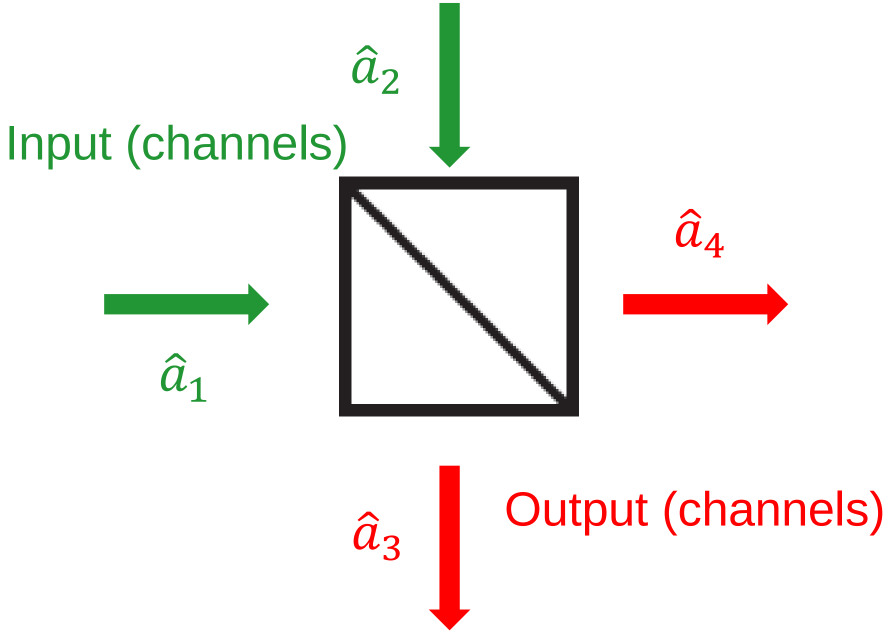
Classical



Quantum

Quantum beam-splitter

Input-ouput relations:



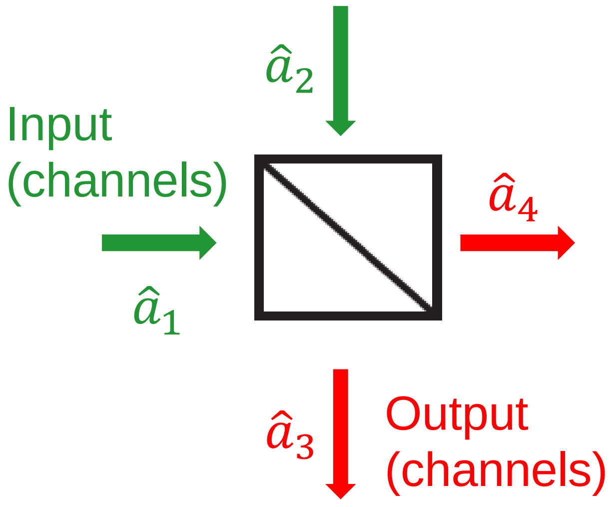
$$\begin{pmatrix} \hat{a}_3 \\ \hat{a}_4 \end{pmatrix} = \overbrace{\begin{pmatrix} R & T \\ T & R \end{pmatrix}}^{\overline{\overline{S}}} \begin{pmatrix} \hat{a}_1 \\ \hat{a}_2 \end{pmatrix}$$

$$\hat{a}_3 = R\hat{a}_1 + T\hat{a}_2$$

$$\hat{a}_4 = T\hat{a}_1 + R\hat{a}_2$$

Quantum beam-splitter

Let us find $\hat{a}_1$ and $\hat{a}_2$ in terms of $\hat{a}_3$ and $\hat{a}_4$:



$$\hat{a}_3 = R\hat{a}_1 + T\hat{a}_2$$

$$\hat{a}_4 = T\hat{a}_1 + R\hat{a}_2$$

$$\bar{\bar{S}}$$

$$\begin{pmatrix} \hat{a}_3 \\ \hat{a}_4 \end{pmatrix} = \begin{pmatrix} R & T \\ T & R \end{pmatrix} \begin{pmatrix} \hat{a}_1 \\ \hat{a}_2 \end{pmatrix}$$

$$R^*\hat{a}_3 = |R|^2\hat{a}_1 + R^*T\hat{a}_2$$

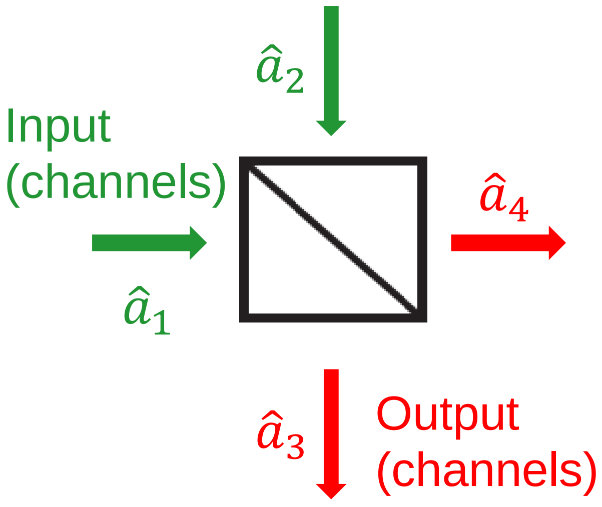
$$T^*\hat{a}_4 = |T|^2\hat{a}_1 + T^*R\hat{a}_2$$

$$|R|^2 + |T|^2 = 1$$

$$R^*T + T^*R = 0$$

$$(|R|^2 + |T|^2)\,\hat{a}_1 + (R^*T + T^*R)\hat{a}_2 = R^*\hat{a}_3 + T^*\hat{a}_4 \;\Rightarrow\; \hat{a}_1 = R^*\hat{a}_3 + T^*\hat{a}_4$$

Quantum beam-splitter



$$\hat{a}_1 = R^*\hat{a}_3 + T^*\hat{a}_4 \qquad \hat{a}_3 = R\hat{a}_1 + T\hat{a}_2$$

$$\hat{a}_2 = T^*\hat{a}_3 + R^*\hat{a}_4 \qquad \hat{a}_4 = T\hat{a}_1 + R\hat{a}_2$$

$$\begin{pmatrix} \hat{a}_3 \\ \hat{a}_4 \end{pmatrix} = \overset{\overline{\overline{S}}}{\begin{pmatrix} R & T \\ T & R \end{pmatrix}} \begin{pmatrix} \hat{a}_1 \\ \hat{a}_2 \end{pmatrix}$$

We can obtain the inverse relations directly by recalling that :

$$\begin{pmatrix} E_3 \\ E_4 \end{pmatrix} = \overline{\overline{S}} \begin{pmatrix} E_1 \\ E_2 \end{pmatrix} \qquad \begin{pmatrix} E_1 \\ E_2 \end{pmatrix} = \overline{\overline{S}}^{-1} \begin{pmatrix} E_3 \\ E_4 \end{pmatrix} = \overline{\overline{S}}^{\dagger} \begin{pmatrix} E_3 \\ E_4 \end{pmatrix}$$

$$|R|^2 + |T|^2 = 1$$

$$\overline{\overline{S}} = \begin{pmatrix} R & T \\ T & R \end{pmatrix} \qquad \overline{\overline{S}}^{-1} = \overline{\overline{S}}^{\dagger} = \begin{pmatrix} R^* & T^* \\ T^* & R^* \end{pmatrix}$$

$$R^*T + T^*R = 0$$

Creation and destruction transformations

$|R|^2 + |T|^2 = 1$

$$\hat{a}_a = R\hat{a}_1 + T\hat{a}_2$$

$$\hat{a}_b = T\hat{a}_1 + R\hat{a}_2$$

$\mathbb{S}$

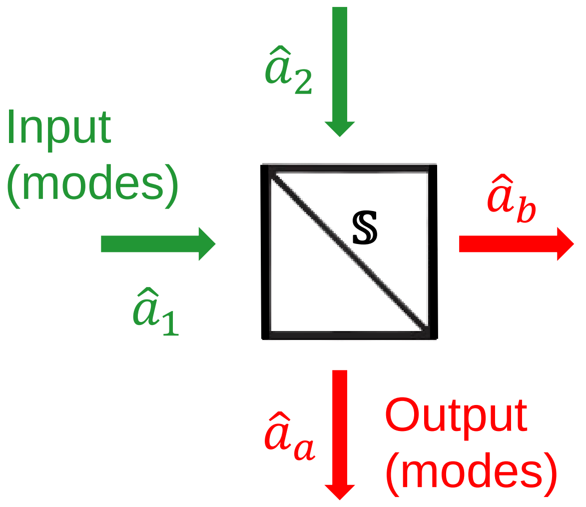
$$\begin{pmatrix}\hat{a}_a\\ \hat{a}_b\end{pmatrix} = \begin{pmatrix}R & T\\ T & R\end{pmatrix}\begin{pmatrix}\hat{a}_1\\ \hat{a}_2\end{pmatrix}$$

$$\hat{a}_1 = R^*\hat{a}_a + T^*\hat{a}_b$$

$$\hat{a}_2 = T^*\hat{a}_a + R^*\hat{a}_b$$

$\mathbb{S}^\dagger = \mathbb{S}^{-1}$

$$\begin{pmatrix}\hat{a}_1\\ \hat{a}_2\end{pmatrix} = \begin{pmatrix}R^* & T^*\\ T^* & R^*\end{pmatrix}\begin{pmatrix}\hat{a}_a\\ \hat{a}_b\end{pmatrix}$$



$$\hat{a}_1^\dagger = R\hat{a}_a^\dagger + T\hat{a}_b^\dagger$$

$$\hat{a}_2^\dagger = T\hat{a}_a^\dagger + R\hat{a}_b^\dagger$$

$\mathbb{S}$

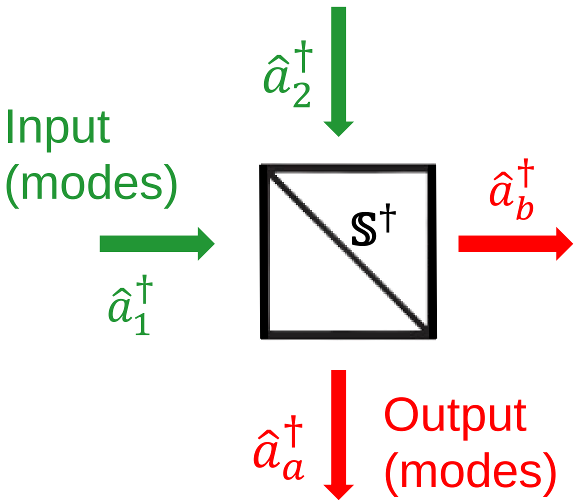
$$\begin{pmatrix}\hat{a}_1^\dagger\\ \hat{a}_2^\dagger\end{pmatrix} = \begin{pmatrix}R & T\\ T & R\end{pmatrix}\begin{pmatrix}\hat{a}_a^\dagger\\ \hat{a}_b^\dagger\end{pmatrix}$$

$$\hat{a}_a^\dagger = R^*\hat{a}_1^\dagger + T^*\hat{a}_2^\dagger$$

$$\hat{a}_b^\dagger = T^*\hat{a}_1^\dagger + R^*\hat{a}_2^\dagger$$

$\mathbb{S}^\dagger = \mathbb{S}^{-1}$

$$\begin{pmatrix}\hat{a}_a^\dagger\\ \hat{a}_b^\dagger\end{pmatrix} = \begin{pmatrix}R^* & T^*\\ T^* & R^*\end{pmatrix}\begin{pmatrix}\hat{a}_1^\dagger\\ \hat{a}_2^\dagger\end{pmatrix}$$

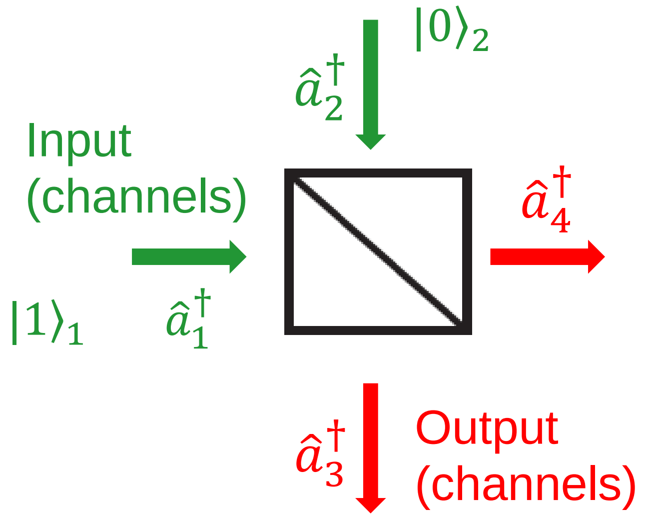


Single photon on a BS

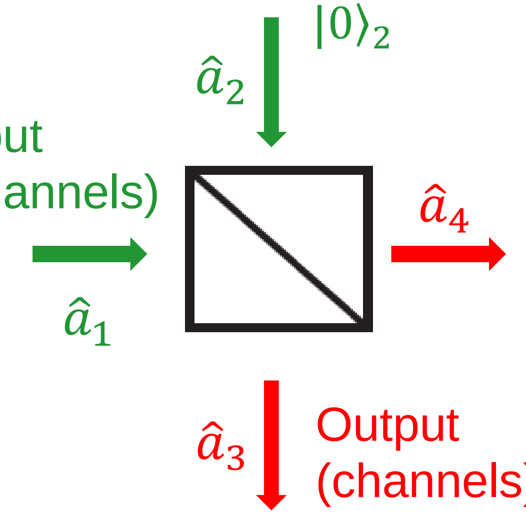
Input State

$|1\rangle_1|0\rangle_2$ (shorthand for $|1\rangle_1|0\rangle_2|0\rangle_3\ |0\rangle_4$)

$$|1\rangle_1|0\rangle_2 = \hat{a}_1^\dagger|0\rangle_1|0\rangle_2$$



Single photon on a BS



Input state

$|1\rangle_1|0\rangle_2$ (shorthand for $|1\rangle_1|0\rangle_2|0\rangle_3|0\rangle_4$)

$$|1\rangle_1|0\rangle_2 = \hat{a}_1^\dagger|0\rangle_1|0\rangle_2 = \hat{a}_1^\dagger|0\rangle_1|0\rangle_2|0\rangle_3|0\rangle_4 = \hat{a}_1^\dagger\,|0\rangle$$

Output state:

$$\hat{a}_1^\dagger = R\hat{a}_3^\dagger + T\hat{a}_4^\dagger$$

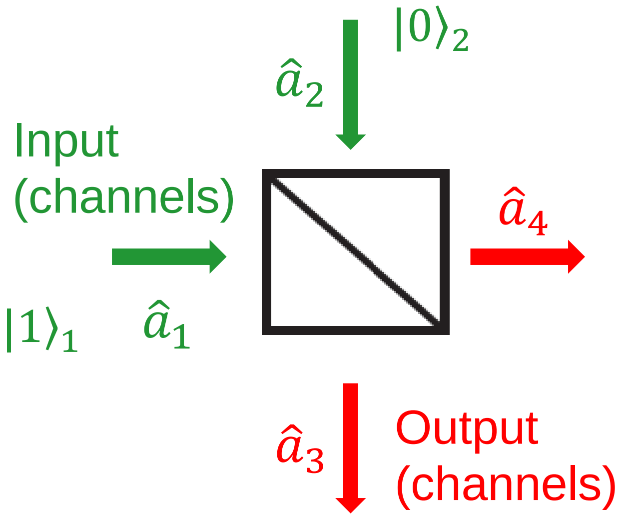
$$\hat{a}_1^\dagger|0\rangle_1|0\rangle_2 = \left(R\hat{a}_3^\dagger + T\hat{a}_4^\dagger\right)|0\rangle_3|0\rangle_4 = \left(R\hat{a}_3^\dagger + T\hat{a}_4^\dagger\right)|0\rangle$$

$$\hat{a}_1^\dagger|0\rangle = R\hat{a}_3^\dagger|0\rangle + T\hat{a}_4^\dagger|0\rangle = R|1\rangle_3|0\rangle_4 + T|0\rangle_3|1\rangle_4$$

Entangled state ! ?

Single photon on a 50/50 BS

Input product state: $|1\rangle_1|0\rangle_2$

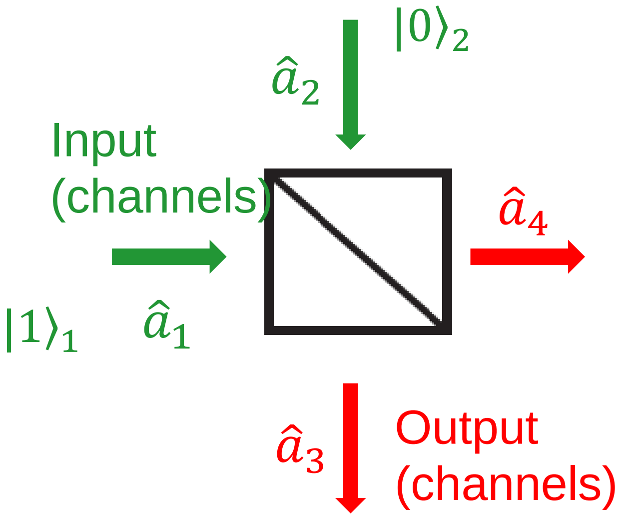


Output entangled state:

$$\hat{a}_1^\dagger = r\hat{a}_3^\dagger + t\hat{a}_4^\dagger$$

$$\hat{a}_1^\dagger|0\rangle_1|0\rangle_2 \Rightarrow \frac{1}{\sqrt{2}}(i|1\rangle_3|0\rangle_4 + |0\rangle_3|1\rangle_4)$$

Single photon on a 50/50 BS



Average output photon number: $\hat{a}_3^\dagger = R^*\hat{a}_1^\dagger + T^*\hat{a}_2^\dagger$

$$\langle \hat{N}_3 \rangle = \langle \hat{a}_3^\dagger \hat{a}_3 \rangle = {}_2\langle 0|\, {}_1\langle 1|\hat{a}_3^\dagger \hat{a}_3|1\rangle_1 |0\rangle_2$$

$$= {}_2\langle 0|\, {}_1\langle 1|\left(R^*\hat{a}_1^\dagger + T^*\hat{a}_2^\dagger\right)(R\hat{a}_1 + T\hat{a}_2)|1\rangle_1 |0\rangle_2$$

$$= |R|^2\, {}_2\langle 0|\, {}_1\langle 0||0\rangle_1 |0\rangle_2$$

$$= |R|^2 \to \frac{1}{2}$$

Likewise one can show that $\langle \hat{n}_4 \rangle = |T|^2 \to \frac{1}{2}$

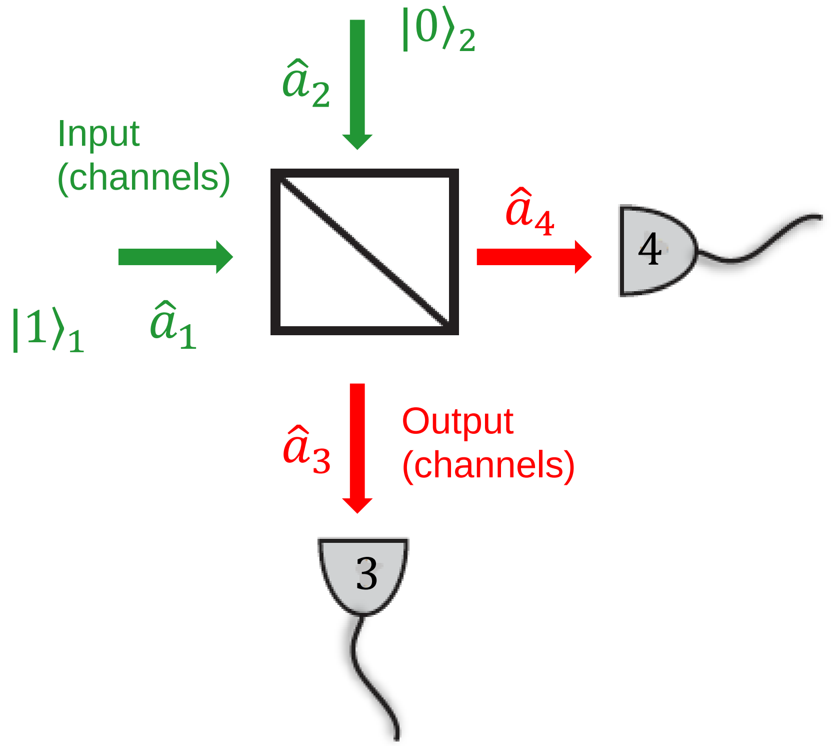
Single photon on a 50/50 BS

Correlations:

$$\langle \hat{N}_3 \hat{N}_4 \rangle = \ {}_2\langle 0|\ {}_1\langle 1|\hat{N}_3\hat{N}_4|1\rangle_1|0\rangle_2$$

$$= \frac{1}{2}\left(\ {}_4\langle 1|\ {}_3\langle 0| + i\ {}_4\langle 0|\ {}_3\langle 1|\right)\hat{N}_3\hat{N}_4\left(i|1\rangle_3|0\rangle_4 + |0\rangle_3|1\rangle_4\right)$$

$= 0$ Result impossible to predict by a classical theory !

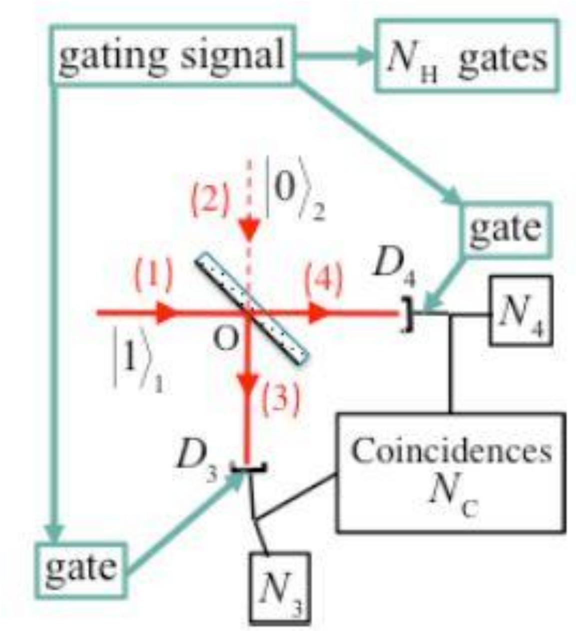


Highly nonclassical correlations between detection in output due to entanglement !

Anti-coincidence measurements were used by Alain Aspect in the 1980's to prove that he had obtained true 1-photon sources of light !

1-photon sources

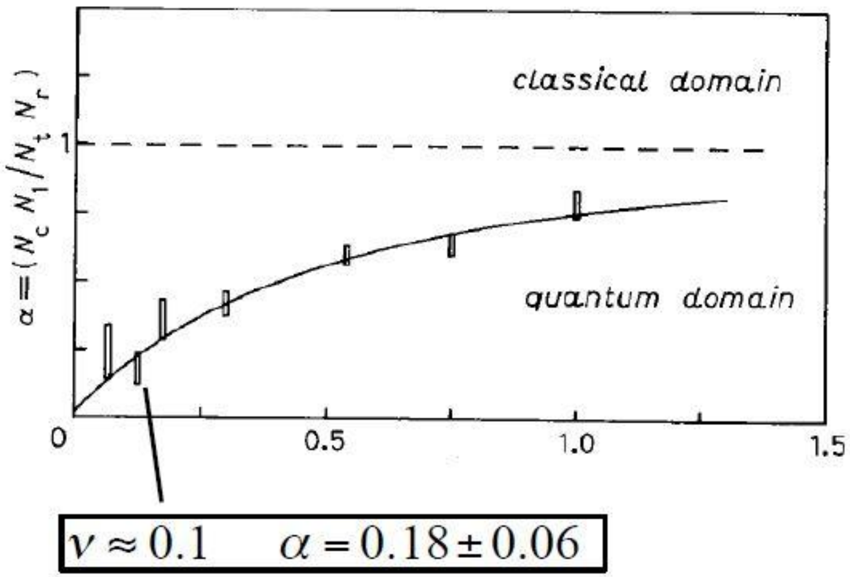
(Wave particle duality, entanglement, Bell inequalities, quantum cryptography,…)



P. Grangier G. Roger and A. Aspect

EUROPHYSICS LETTERS

Europhys. Lett., **1** (4), pp. 173-179 (1986)

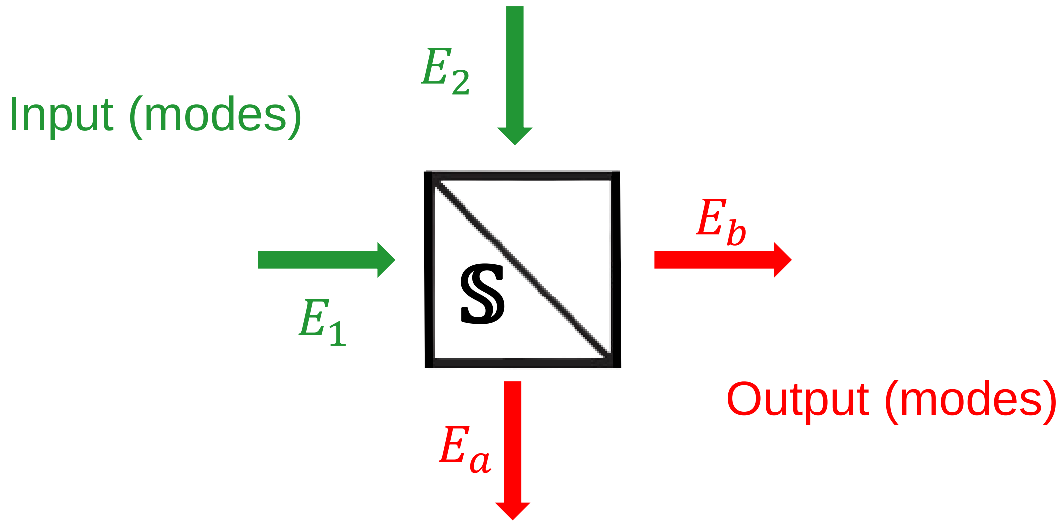


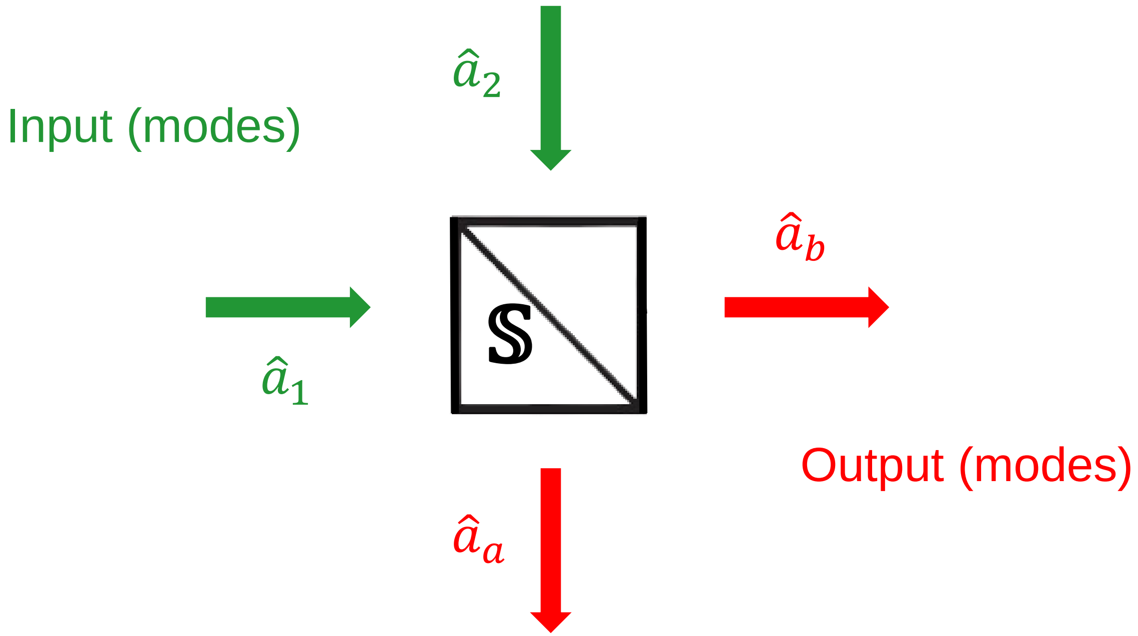
Alain Aspect : https://www.coursera.org/learn/quantum-optics-single-photon

The Quantum beam-splitter

Operator transformations in the quantum beam-splitter

Input-output relations:





Classical

$$\begin{pmatrix} E_a \\ E_b \end{pmatrix} = \begin{pmatrix} R & T \\ T & R \end{pmatrix} \begin{pmatrix} E_1 \\ E_2 \end{pmatrix}$$

$$|R|^2 + |T|^2 = 1$$

Quantum

$$\begin{pmatrix} \hat{a}_a \\ \hat{a}_b \end{pmatrix} = \begin{pmatrix} R & T \\ T & R \end{pmatrix} \begin{pmatrix} \hat{a}_1 \\ \hat{a}_2 \end{pmatrix}$$

Creation and destruction transformations

$$|R|^2 + |T|^2 = 1$$

$$\hat{a}_a = R\hat{a}_1 + T\hat{a}_2$$

$$\hat{a}_b = T\hat{a}_1 + R\hat{a}_2$$

$\mathbb{S}$

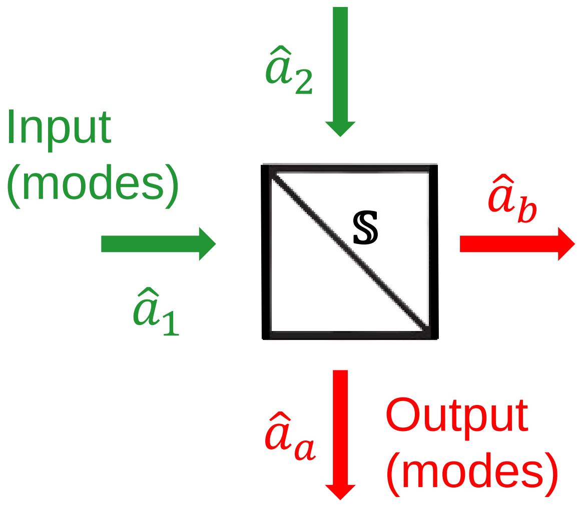
$$\begin{pmatrix} \hat{a}_a \\ \hat{a}_b \end{pmatrix} = \begin{pmatrix} R & T \\ T & R \end{pmatrix} \begin{pmatrix} \hat{a}_1 \\ \hat{a}_2 \end{pmatrix}$$

$$\hat{a}_1 = R^*\hat{a}_a + T^*\hat{a}_b$$

$$\hat{a}_2 = T^*\hat{a}_a + R^*\hat{a}_b$$

$\mathbb{S}^\dagger$

$$\begin{pmatrix} \hat{a}_1 \\ \hat{a}_2 \end{pmatrix} = \begin{pmatrix} R^* & T^* \\ T^* & R^* \end{pmatrix} \begin{pmatrix} \hat{a}_a \\ \hat{a}_b \end{pmatrix}$$



$$\hat{a}_1^\dagger = R\hat{a}_a^\dagger + T\hat{a}_b^\dagger$$

$$\hat{a}_2^\dagger = T\hat{a}_a^\dagger + R\hat{a}_b^\dagger$$

$\mathbb{S}$

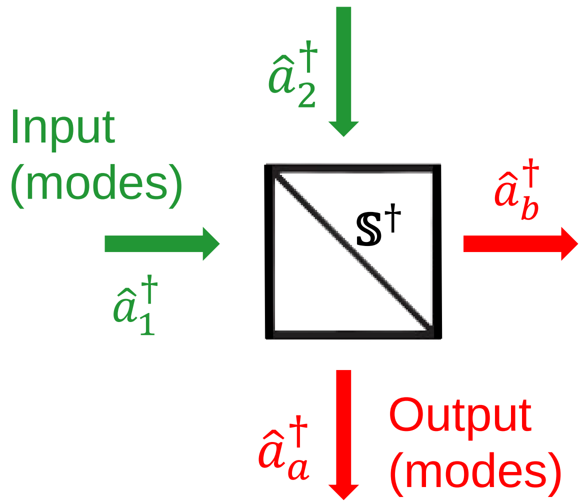
$$\begin{pmatrix} \hat{a}_1^\dagger \\ \hat{a}_2^\dagger \end{pmatrix} = \begin{pmatrix} R & T \\ T & R \end{pmatrix} \begin{pmatrix} \hat{a}_a^\dagger \\ \hat{a}_b^\dagger \end{pmatrix}$$

$$\hat{a}_a^\dagger = R^*\hat{a}_1^\dagger + T^*\hat{a}_2^\dagger$$

$$\hat{a}_b^\dagger = T^*\hat{a}_1^\dagger + R^*\hat{a}_2^\dagger$$

$\mathbb{S}^\dagger$

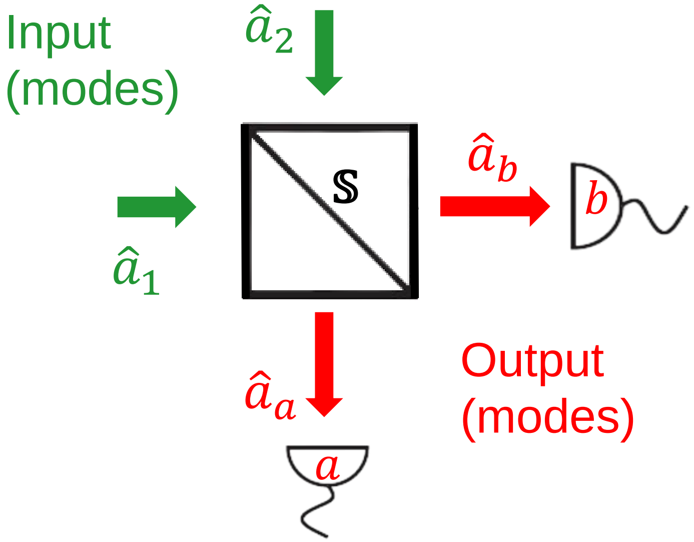
$$\begin{pmatrix} \hat{a}_a^\dagger \\ \hat{a}_b^\dagger \end{pmatrix} = \begin{pmatrix} R^* & T^* \\ T^* & R^* \end{pmatrix} \begin{pmatrix} \hat{a}_1^\dagger \\ \hat{a}_2^\dagger \end{pmatrix}$$



Single photon on a BS (Fock space) : "normal ordering"

Input representation :

$$|\Psi\rangle = |1\rangle_1|0\rangle_2 = \hat{a}_1^\dagger|0\rangle_1|0\rangle_2 = \hat{a}_1^\dagger|0\rangle$$



$s\left(\mathcal{E}_{\text{p.w.}}^{(1)}\right)^2 \rightarrow \eta$

s : sensitivity of the detector :

η : Quantum efficiency of the detector ($\eta \leq 1$)

$$w^{(1)}(\boldsymbol{r}_a, t) \equiv s\left\|\widehat{\boldsymbol{E}}^{(+)}(\boldsymbol{r}_a, t)|\Psi\rangle\right\|^2 = s\langle\Psi|\widehat{\boldsymbol{E}}^{(-)}(\boldsymbol{r}_a, t)\widehat{\boldsymbol{E}}^{(+)}(\boldsymbol{r}_a, t)|\Psi\rangle = s\left(\mathcal{E}_{\text{p.w.}}^{(1)}\right)^2\langle\Psi|\hat{a}_a^\dagger\hat{a}_a|\Psi\rangle = \eta\langle 0|\hat{a}_1\hat{a}_a^\dagger\hat{a}_a\hat{a}_1^\dagger|0\rangle$$

$$\hat{a}_a = R\hat{a}_1 + T\hat{a}_2 \qquad\qquad \hat{a}_a^\dagger = R^*\hat{a}_1^\dagger + T^*\hat{a}_2^\dagger$$

$$= \eta\langle 0|\hat{a}_1\hat{a}_a^\dagger\hat{a}_a\hat{a}_1^\dagger|0\rangle = \eta\left\langle 0\middle|\hat{a}_1\left(R^*\hat{a}_1^\dagger + T^*\hat{a}_2^\dagger\right)\left(R\hat{a}_1 + T\hat{a}_2\right)\hat{a}_1^\dagger\middle|0\right\rangle = \eta|R|^2\left\langle 0\middle|\hat{a}_1\hat{a}_1^\dagger\hat{a}_1\hat{a}_1^\dagger\middle|0\right\rangle \qquad \left[\hat{a}_1, \hat{a}_1^\dagger\right] = 1$$

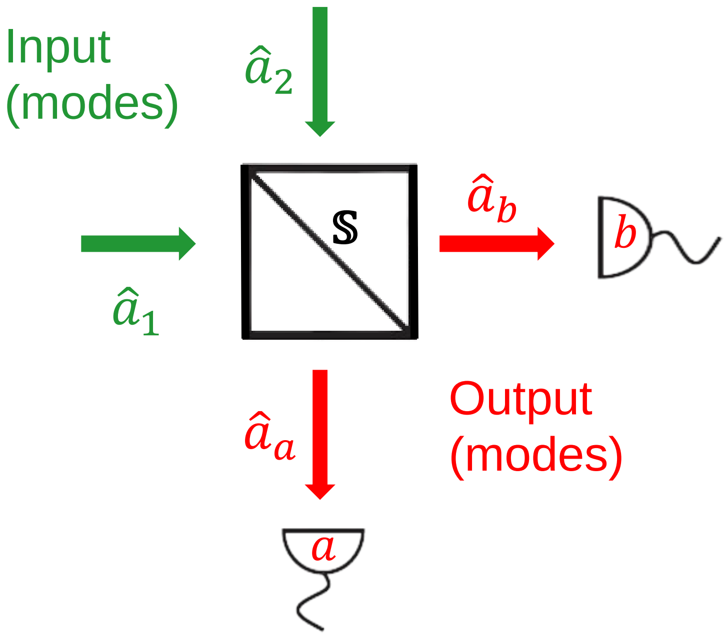
$$= \eta|R|^2\langle 0|\left(\hat{a}_1^\dagger\hat{a}_1 - 1\right)\left(\hat{a}_1^\dagger\hat{a}_1 - 1\right)|0\rangle = \eta|R|^2\langle 0|\hat{a}_1^\dagger\hat{a}_1\hat{a}_1^\dagger\hat{a}_1|0\rangle - 2\eta|R|^2\langle 0|\hat{a}_1^\dagger\hat{a}|0\rangle + \eta|R|^2\langle 0|0\rangle$$

$$= \eta|R|^2\langle 0|\hat{a}_1^\dagger\hat{a}_1^\dagger\hat{a}_1\hat{a}_1|0\rangle - \eta|R|^2\langle 0|\hat{a}_1^\dagger\hat{a}|0\rangle + \eta|R|^2\langle 0|0\rangle = \eta|R|^2$$

Single photon detection probabilities follow semi-classical predictions

Input representation :

$$|\Psi\rangle = |1\rangle_1|0\rangle_2 = \hat{a}_1^\dagger|0\rangle_1|0\rangle_2 \equiv \hat{a}_1^\dagger|0\rangle$$

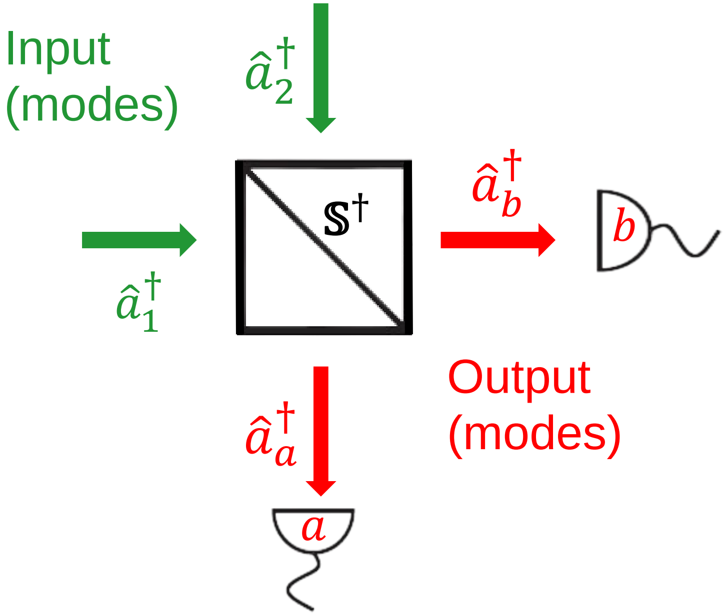


$$w^{(1)}(\boldsymbol{r}_a, t) \equiv s\left\|\widehat{\boldsymbol{E}}^{(+)}(\boldsymbol{r}_a, t)|\Psi\rangle\right\|^2 = \eta|R|^2$$

$$w^{(1)}(\boldsymbol{r}_b, t) \equiv s\left\|\widehat{\boldsymbol{E}}^{(+)}(\boldsymbol{r}_b, t)|\Psi\rangle\right\|^2 = \eta|T|^2$$

1-photon incidence is beyond the "classical" regime

$$|\Psi\rangle = |1\rangle_1|0\rangle_2 = \hat{a}_1^\dagger|0\rangle = R\hat{a}_a^\dagger|0\rangle + T\hat{a}_b^\dagger|0\rangle$$



$$w^{(2)}(\boldsymbol{r}_a, t; \boldsymbol{r}_b, t') \equiv s^2 \left\| \frac{1}{\sqrt{2}} \widehat{\boldsymbol{E}}^{(+)}(\boldsymbol{r}_a, t)\widehat{\boldsymbol{E}}^{(+)}(\boldsymbol{r}_b, t')|\Psi\rangle \right\|^2$$

$$= \frac{\eta^2}{2} \left\| (\hat{a}_a\hat{a}_b)\left(R\hat{a}_a^\dagger|0\rangle + T\hat{a}_b^\dagger|0\rangle\right) \right\|^2 = 0$$

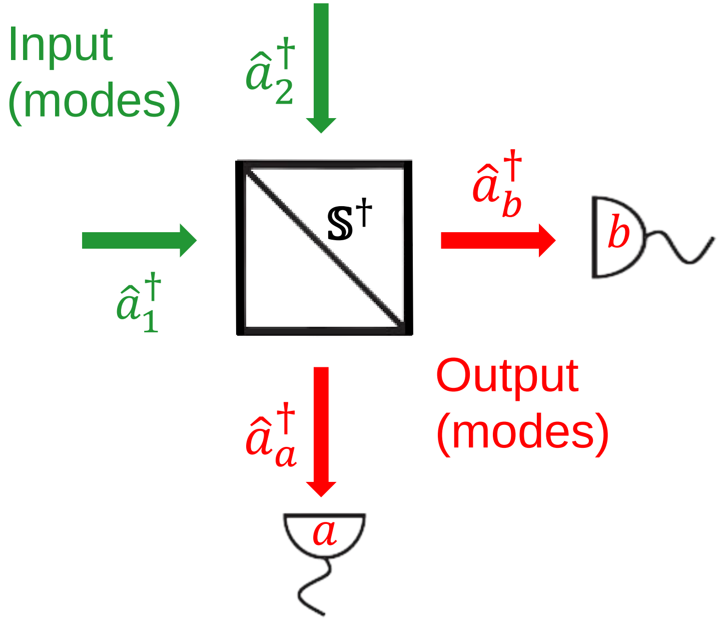
Generally considered as proof that light is indeed quantum !

But also, … spooky "action" at a distance !

1-photon on a beam-splitter as a means of “entanglement” ?

Input representation :

$$|\Psi\rangle = |1\rangle_1|0\rangle_2 = \hat{a}_1^\dagger|0\rangle_1|0\rangle_2 = \hat{a}_1^\dagger|0\rangle$$



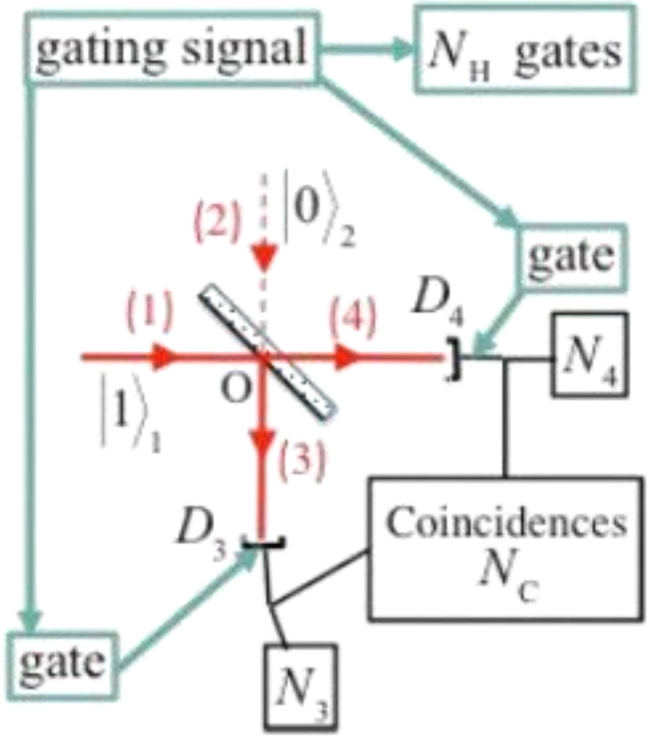
Output representation: $\hat{a}_1^\dagger = R\hat{a}_a^\dagger + T\hat{a}_b^\dagger$

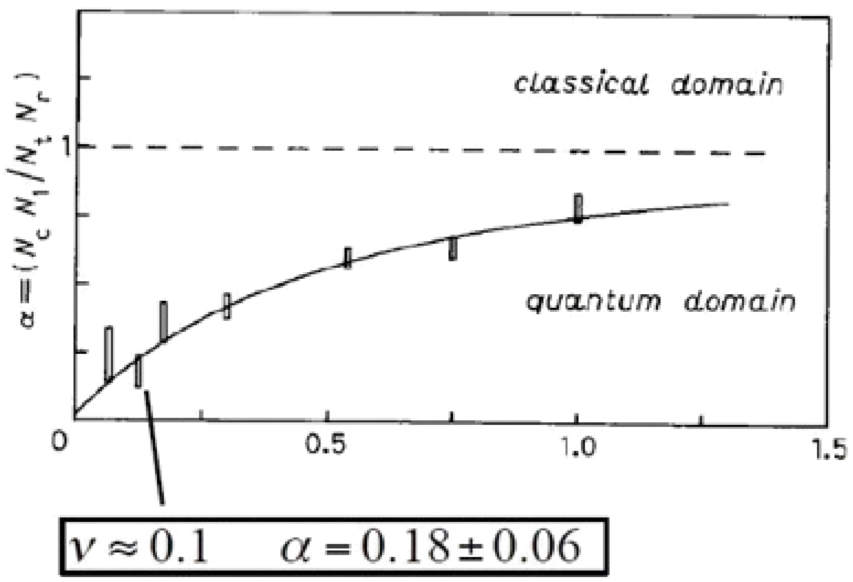
$$|\Psi\rangle = \hat{a}_1^\dagger|0\rangle = R\hat{a}_a^\dagger|0\rangle + T\hat{a}_b^\dagger|0\rangle = R|1\rangle_a|0\rangle_b + T|0\rangle_a|1\rangle_b$$

Entangled state ?

Since the 1980's we know that light obeys quantum mechanical laws!

1-photon sources





"Quantum phenomena do not occur in a Hilbert space. They occur in a laboratory." Asher Peres

Alain Aspect : https://www.coursera.org/learn/quantum-optics-single-photon

Number state on a BS

Number state on a beam-splitter $|n\rangle_1|0\rangle_2$

$$|n\rangle_1|0\rangle_2 = \frac{\left(\hat{a}_1^\dagger\right)^n}{\sqrt{n!}}|0\rangle$$

$$= \frac{\left(R\hat{a}_3^\dagger + T\hat{a}_4^\dagger\right)^n}{\sqrt{n!}}|0\rangle = \frac{1}{\sqrt{n!}}\sum_{k=0}^{n}\binom{n}{k}\left(R\hat{a}_3^\dagger\right)^k\left(T\hat{a}_4^\dagger\right)^{n-k}|0\rangle \qquad \hat{a}_1^\dagger = R\hat{a}_3^\dagger + T\hat{a}_4^\dagger$$

$$= \frac{1}{\sqrt{n!}}\sum_{k=0}^{n}\binom{n}{k}R^k T^{n-k}\left(\hat{a}_3^\dagger\right)^k\left(\hat{a}_4^\dagger\right)^{n-k}|0\rangle$$

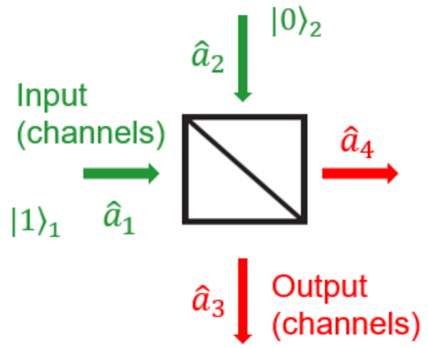
Binomial coefficient $\binom{n}{k} = \frac{n!}{k!\,(n-k)!}$

$$= \sum_{k=0}^{n} R^k T^{n-k}\sqrt{\binom{n}{k}}\,|k\rangle_3|n-k\rangle_4$$

The probability of finding a photon number $|j\rangle_3$ at in the output state of detector 3

$$P(j_3) = |\ {}_4\langle n-j|_3\langle j|n\rangle_1|0\rangle_2\,|^2 = R^{2j}T^{2(n-j)}\binom{n}{j}$$

Coherent state on a BS



Input product state of a coherent state in channel 1 : $|\alpha\rangle_1|0\rangle_2$

$$|\alpha\rangle_1|0\rangle_2 = e^{\alpha\hat{a}_1^\dagger-\alpha^*\hat{a}_1}|0\rangle = e^{\alpha\hat{a}_1^\dagger-\alpha^*\hat{a}_1}\ |0\rangle$$

$$= e^{\alpha\left(R\hat{a}_3^\dagger+T\hat{a}_4^\dagger\right)-\alpha^*(R^*\hat{a}_3+T^*\hat{a}_4)}\ |0\rangle$$

$$= e^{\left(R\alpha\hat{a}_3^\dagger+T\alpha\hat{a}_4^\dagger\right)-(R^*\alpha^*\hat{a}_3+T^*\alpha^*\hat{a}_4)}\ |0\rangle$$

$$= e^{R\alpha\hat{a}_3^\dagger-R^*\alpha^*\hat{a}_3}e^{T\alpha\hat{a}_4^\dagger-T^*\alpha^*\hat{a}_4}|0\rangle$$

$$= |R\alpha\rangle_3|T\alpha\rangle_4$$

$$\hat{a}_1 = R^*\hat{a}_3 + T^*\hat{a}_4$$

$$\hat{a}_1^\dagger = R\hat{a}_3^\dagger + T\hat{a}_4^\dagger$$

For coherent states, both input and output state descriptions are product states

There are other ways to obtain this result. Can you find them ?

Quadrature operators

Quadrature parameters - Classical Considerations

Classical electromagnetic field: $E(t) = \mathcal{E}_0 \sin(\boldsymbol{k}\cdot\boldsymbol{r})\sin(\omega t+\phi) = \mathcal{E}_0[\sin(\phi)\cos(\omega t)+\cos(\phi)\sin(\omega t)]$
(quadrature description)

$$= \mathcal{E}_0 \sin(\boldsymbol{k}\cdot\boldsymbol{r})\,[X_1\cos(\omega t)+X_2\sin(\omega t)]$$

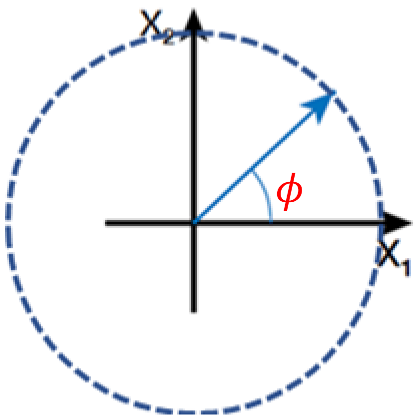
Quadrature parameters : $X_1 = \sin\phi$ $\quad X_2 = \cos\phi$

Phasor representation of the field:

$$E(t) = \mathcal{E}_0\alpha e^{-i\omega t} + \text{c.c.}$$

$$= \mathcal{E}_0\left[\alpha e^{-i\omega t} + \alpha^* e^{i\omega t}\right]$$

$$= \mathcal{E}_0[X_1\cos(\omega t)+X_2\sin(\omega t)]$$



$$\alpha = |\alpha|e^{i\phi} = (X_1 + iX_2)$$

$$X_1 = \text{Re}(\alpha) = \frac{1}{2}(\alpha+\alpha^*) \qquad X_2 = \text{Im}(\alpha) = \frac{1}{2i}(\alpha-\alpha^*)$$

Quantum Quadrature operators

$$\hat{H} = \hbar\omega\left(\hat{X}_1^2 + \hat{X}_2^2\right) = \frac{\omega}{2}\left(\hat{Q}^2 + \hat{P}^2\right)$$

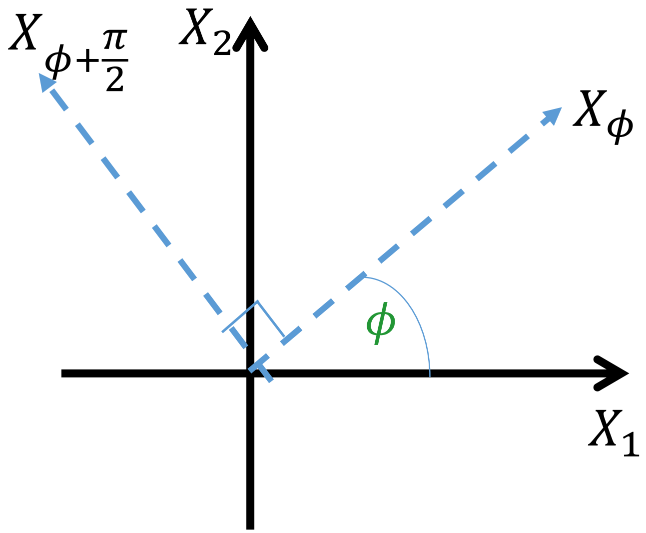
Definition: Quadrature Operators

$$\hat{X}_1 \equiv \frac{1}{2}\left(\hat{a} + \hat{a}^\dagger\right) \equiv \frac{1}{\sqrt{2\hbar}}\hat{Q}$$

$$\hat{X}_2 \equiv \frac{1}{2i}\left(\hat{a} - \hat{a}^\dagger\right) \equiv \frac{1}{\sqrt{2\hbar}}\hat{P}$$

$$\hat{X}_\phi = \frac{1}{2}\left(\hat{a}e^{-i\phi} + \hat{a}^\dagger e^{i\phi}\right)$$

$$\hat{X}_{\phi+\frac{\pi}{2}} = \frac{1}{2i}\left(\hat{a}e^{-i\phi} - \hat{a}^\dagger e^{i\phi}\right)$$



Commutation and uncertainty relations:

$$\left[\hat{X}_1, \hat{X}_2\right] = \frac{i}{2}$$

$$\left[\hat{Q}, \hat{P}\right] = i\hbar$$

$$\left[\hat{X}_\phi, \hat{X}_{\phi+\frac{\pi}{2}}\right] = \frac{i}{2}$$

$$\Delta X_1 \Delta X_2 \geq \frac{1}{4}$$

$$\Delta Q \Delta P \geq \frac{\hbar}{2}$$

$$\Delta X_\varphi \Delta X_{\varphi+\frac{\pi}{2}} \geq \frac{1}{4}$$

Relations between dephased Quadrature operators

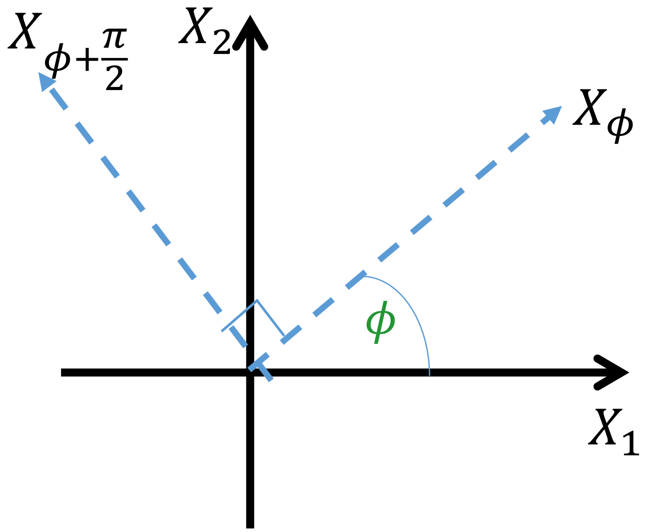
Definition: Quadrature Operators

$$\hat{X}_1 \equiv \frac{1}{2}(\hat{a} + \hat{a}^\dagger)$$

$$\hat{X}_2 \equiv \frac{1}{2i}(\hat{a} - \hat{a}^\dagger)$$

$$\hat{a} = \hat{X}_1 + i\hat{X}_2$$

$$\hat{a}^\dagger = \hat{X}_1 - i\hat{X}_2$$



$$\hat{X}_\phi = \frac{1}{2}(\hat{a}e^{-i\phi} + \hat{a}^\dagger e^{i\phi}) = \left[\hat{X}_1 \frac{(e^{-i\phi} + e^{i\phi})}{2} + \hat{X}_2 \left(\frac{e^{i\phi} - e^{-i\phi}}{2i}\right)\right] = \hat{X}_1 \cos\phi + \hat{X}_2 \sin\phi$$

$$\hat{X}_{\phi+\frac{\pi}{2}} = \frac{1}{2i}(\hat{a}e^{-i\phi} - \hat{a}^\dagger e^{i\phi}) = \left(\hat{X}_1 \frac{(e^{-i\phi} - e^{i\phi})}{2i} + i\hat{X}_2 \left(\frac{e^{-i\phi} + e^{i\phi}}{2i}\right)\right) = -\hat{X}_1 \sin\phi + \hat{X}_2 \cos\phi$$

Field operator $\hat{\boldsymbol{E}}_\ell(\chi_\ell)$ in terms of Quadrature

$$\hat{X}_\phi = \frac{1}{2}\left(\hat{a}e^{-i\phi} + \hat{a}^\dagger e^{i\phi}\right)$$

$$\hat{X}_{\phi+\frac{\pi}{2}} = \frac{1}{2i}\left(\hat{a}e^{-i\phi} - \hat{a}^\dagger e^{i\phi}\right)$$

$$\hat{a} = \hat{X}_\phi e^{i\phi} + i\hat{X}_{\varphi+\frac{\pi}{2}} e^{i\phi}$$

$$\hat{a}^\dagger = \hat{X}_\phi e^{-i\phi} - i\hat{X}_{\phi+\frac{\pi}{2}} e^{-i\phi}$$

For a single mode ℓ : $\hat{\boldsymbol{E}}(\chi) = \boldsymbol{\epsilon}\mathcal{E}^{(1)}_{\text{p.w.}}\left[\hat{a}e^{-i\chi} + \hat{a}^\dagger e^{i\chi}\right]$

$$= \boldsymbol{\epsilon}\mathcal{E}^{(1)}_{\text{p.w.}}\left[\hat{X}_\phi e^{-i(\chi_\ell-\phi)} + \hat{X}_\phi e^{i(\chi_\ell-\phi)} + i\hat{X}_{\phi+\frac{\pi}{2}} e^{-i(\chi_\ell-\phi)} - i\hat{X}_{\phi+\frac{\pi}{2}} e^{i(\chi_\ell-\phi)}\right]$$

$$\hat{\boldsymbol{E}}(\chi) = 2\boldsymbol{\epsilon}\mathcal{E}^{(1)}_{\text{p.w.}}\left[\hat{X}_\phi \cos(\chi_\ell - \phi) + \hat{X}_{\phi+\frac{\pi}{2}} \sin(\chi_\ell - \phi)\right]$$

$$\chi_\ell \equiv \omega_\ell t - \boldsymbol{k}_\ell \cdot \boldsymbol{r} - \frac{\pi}{2}$$

A famous « nuissance » in $\hat{\boldsymbol{E}}_\ell(\chi_\ell)$

$$\chi_\ell \equiv \omega_\ell t - \boldsymbol{k}_\ell \cdot \boldsymbol{r} - \frac{\pi}{2}$$

$$\hat{\boldsymbol{E}}(\chi) = 2\boldsymbol{\epsilon}\mathcal{E}^{(1)}_{\text{p.w.}}\left[\hat{X}_\phi \cos(\chi_\ell - \phi) + \hat{X}_{\phi+\frac{\pi}{2}} \sin(\chi_\ell - \phi)\right]$$

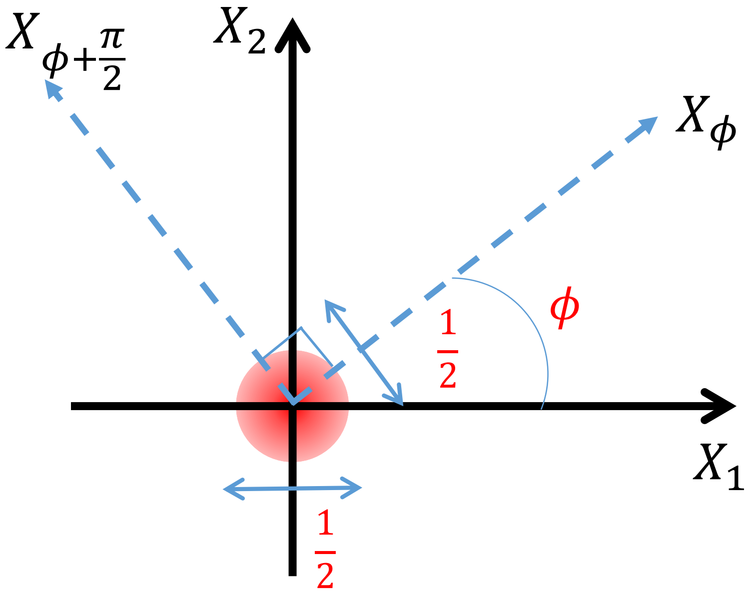
$$= 2\boldsymbol{\epsilon}\mathcal{E}^{(1)}_{\text{p.w.}}\left[\hat{X}_\phi \cos\left(\omega t - \boldsymbol{k}\cdot\boldsymbol{r} - \phi - \frac{\pi}{2}\right) + \hat{X}_{\phi+\frac{\pi}{2}} \sin\left(\omega t - \boldsymbol{k}\cdot\boldsymbol{r} - \phi - \frac{\pi}{2}\right)\right]$$

$$= 2\boldsymbol{\epsilon}\mathcal{E}^{(1)}_{\text{p.w.}}\left[\hat{X}_\phi \sin(\omega t - \boldsymbol{k}\cdot\boldsymbol{r} - \phi) - \hat{X}_{\phi+\frac{\pi}{2}} \cos(\omega t - \boldsymbol{k}\cdot\boldsymbol{r} - \phi)\right]$$

Phase space distribution of the vacuum

Vacuum state has isotropic Gaussian fluctuations

Vacuum state: $|0\rangle$



$$P^{(0)}(X_1) = |\langle X_1|0\rangle|^2 = \frac{e^{-X_1^2}}{\sqrt{\pi}}$$

$$P^{(0)}(X_2) = |\langle X_2|0\rangle|^2 = \frac{e^{-X_2^2}}{\sqrt{\pi}}$$

Fluctuations:

$$\Delta X_1 = \sqrt{\langle 0|\hat{X}_1^2|0\rangle - \langle 0|\hat{X}_1|0\rangle^2} = \frac{1}{2}$$

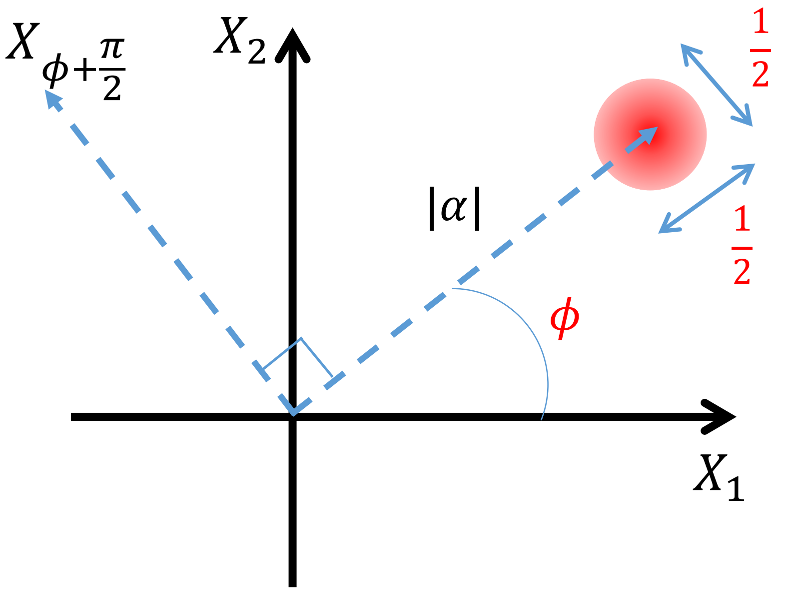
$$\Delta X_\varphi = \sqrt{\langle 0|\hat{X}_\varphi^2|0\rangle - \langle 0|\hat{X}_\varphi|0\rangle^2} = \frac{1}{2}$$

$$\Delta X_2 = \sqrt{\langle 0|\hat{X}_2^2|0\rangle - \langle 0|\hat{X}_2|0\rangle^2} = \frac{1}{2}$$

$$\Delta X_{\varphi+\frac{\pi}{2}} = \sqrt{\left\langle 0|\hat{X}^2_{\varphi+\frac{\pi}{2}}|0\right\rangle - \left\langle 0|\hat{X}_{\varphi+\frac{\pi}{2}}|0\right\rangle^2} = \frac{1}{2}$$

Phase space distribution of coherent states

Coherent states have isotropic Gaussian fluctuations



Coherent state: $|\alpha\rangle$ $\quad\quad \alpha = |\alpha|e^{i\phi} = (X_1 + iX_2)$

$$P^{(0)}(X_1) = |\langle X_1|\alpha\rangle|^2 = \langle X_1|\alpha\rangle = \frac{e^{-(X_1-\mathrm{Re}(\alpha))^2}}{\sqrt{\pi}}$$

$$P^{(0)}(X_2) = |\langle X_2|\alpha\rangle|^2 = \frac{e^{-(X_2-\mathrm{Im}(\alpha))^2}}{\sqrt{\pi}}$$

Fluctuations:

$$\Delta X_1 = \sqrt{\langle\alpha|\hat{X}_1^2|\alpha\rangle - \langle\alpha|\hat{X}_1|\alpha\rangle^2} = \frac{1}{2}$$

$$\Delta X_2 = \sqrt{\langle\alpha|\hat{X}_2^2|\alpha\rangle - \langle\alpha|\hat{X}_2|\alpha\rangle^2} = \frac{1}{2}$$

$$\Delta X_\varphi = \sqrt{\langle\alpha|\hat{X}_\varphi^2|\alpha\rangle - \langle\alpha|\hat{X}_\varphi|\alpha\rangle^2} = \frac{1}{2}$$

$$\Delta X_{\varphi+\frac{\pi}{2}} = \sqrt{\left\langle\alpha|\hat{X}^2_{\varphi+\frac{\pi}{2}}|\alpha\right\rangle - \left\langle\alpha|\hat{X}_{\varphi+\frac{\pi}{2}}|\alpha\right\rangle^2} = \frac{1}{2}$$

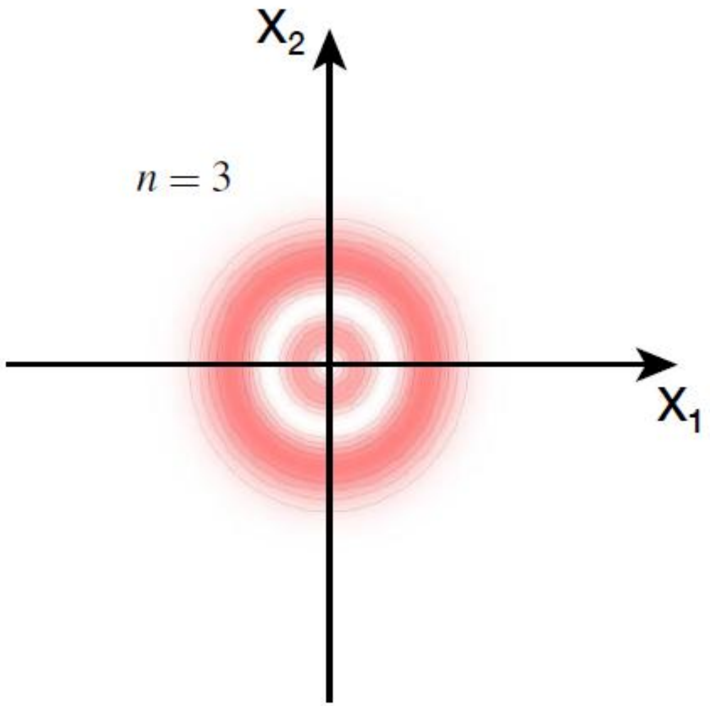
Phase space distribution of field states

Fock state: $|n\rangle$

$H_n(x)$ is a Hermite polynomial of order n

$$P^{(n)}(X_1) = |\langle X_1|n\rangle|^2$$

$$= \sqrt{\frac{1}{\pi}}\frac{1}{n!}e^{-X_1^2}[H_n(X_1)]^2$$



Fluctuations:

$$\Delta X_1 = \sqrt{\langle n|\hat{X}_1^2|n\rangle - \langle n|\hat{X}_1|n\rangle^2} = \frac{\sqrt{2n+1}}{2}$$

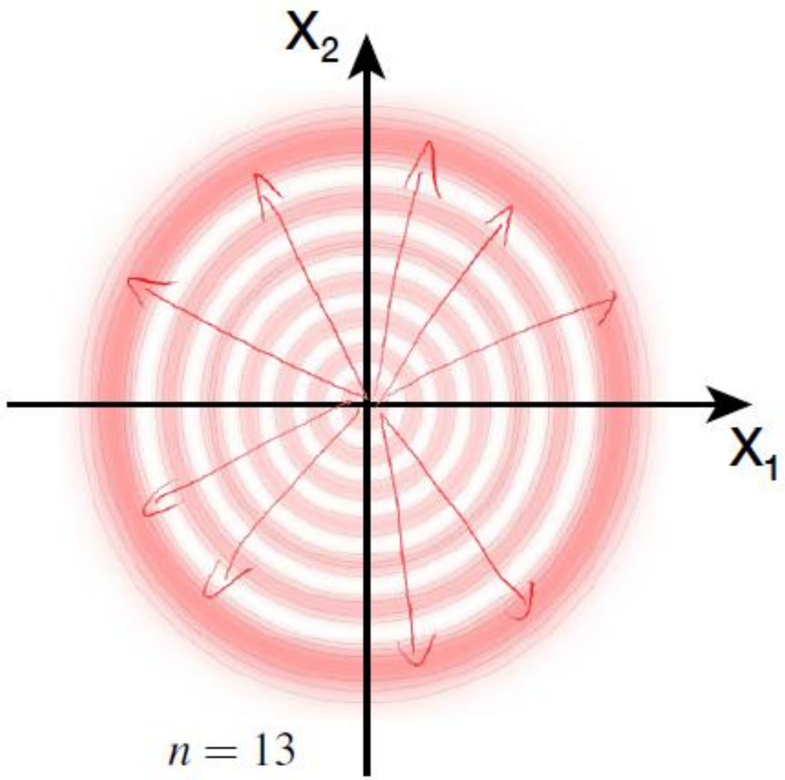
$$\Delta X_2 = \sqrt{\langle n|\hat{X}_2^2|n\rangle - \langle n|\hat{X}_2|n\rangle^2} = \frac{\sqrt{2n+1}}{2}$$

Phase space distribution of field states

Fock state: $|n\rangle$

$$P^{(n)}(X_1) = |\langle X_1|n\rangle|^2$$

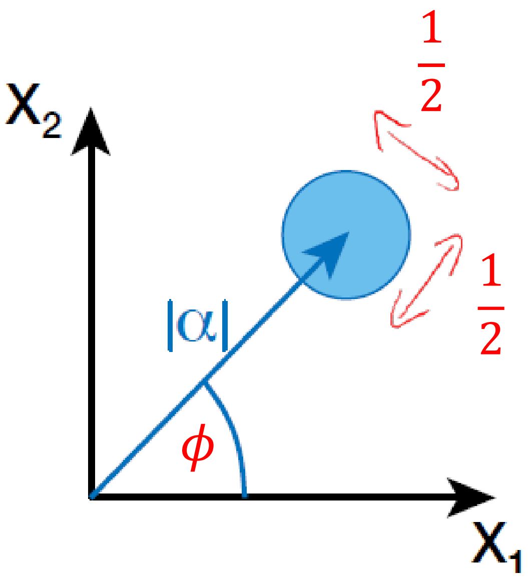
$$= \sqrt{\frac{1}{\pi}\frac{1}{n!}}e^{-X_1^2}[H_n(X_1)]^2$$



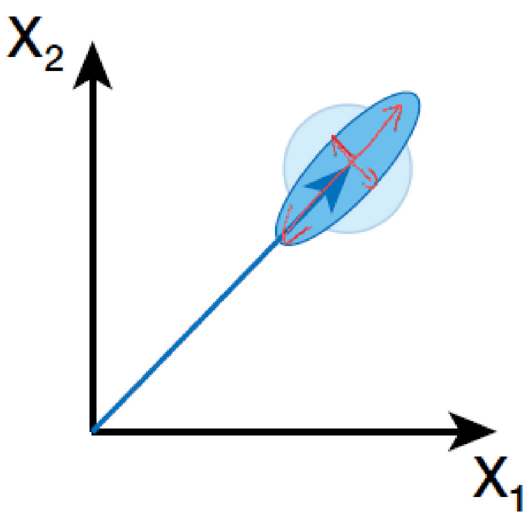
Fluctuations:

$$\Delta X_1 = \sqrt{\langle n|\hat{X}_1^2|n\rangle - \langle n|\hat{X}_1|n\rangle^2} = \frac{\sqrt{2n+1}}{2}$$

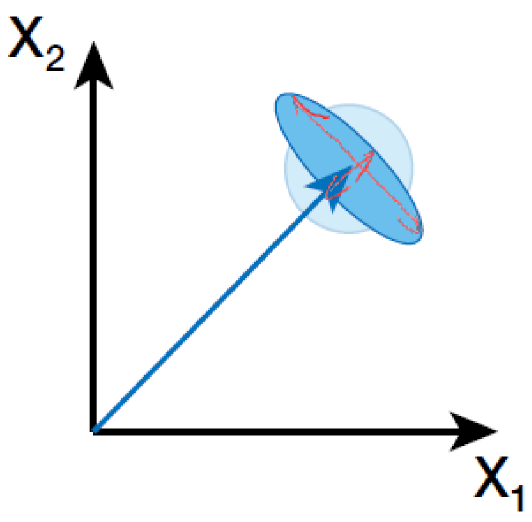
Squeezed states of light



Coherent state



Phase squeezed light



amplitude squeezed light